Here early? Check out the Stardew Valley Boardgame up front!

Please sit towards the center of the room

Normal RVs

CSE 312 Summer 25
Lecture 15

Announcements

Homework 4 due today

Homework 5 out today

Midterm overview at 1:15pm tomorrow in Gates 371

One-on-one office hours

Quiz 5 on Friday

Normal Random Variable

$\mu \quad \sigma^2$

X is a normal (aka Gaussian) random variable with mean μ and variance σ^2 (written $X \sim \mathcal{N}(\mu, \sigma^2)$) if it has the density:

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

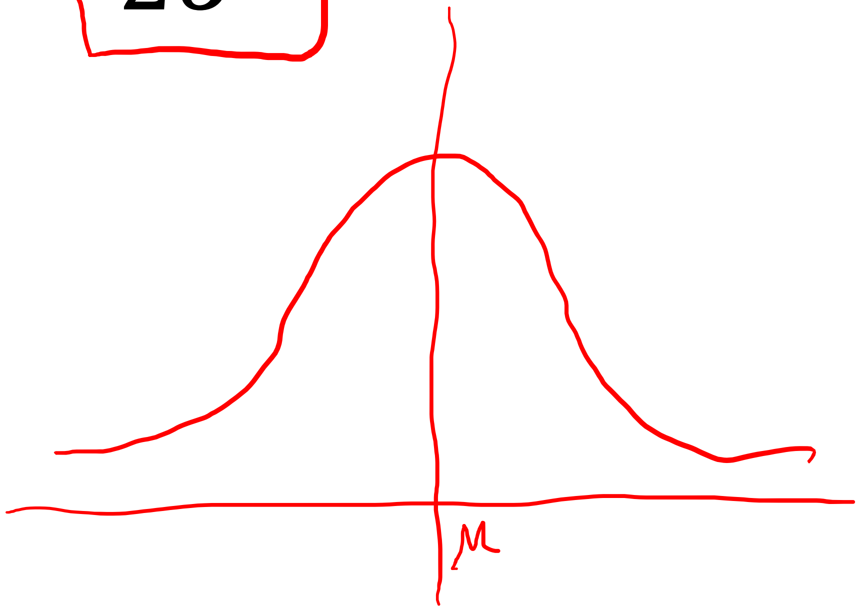
Let's get some intuition for that density...

Is $\mathbb{E}[X] = \mu$?

Yes! Plug in $\mu - k$ and $\mu + k$ and you'll get the same density for every k. The density is symmetric around μ. The expectation must be μ.

Breaking down the density

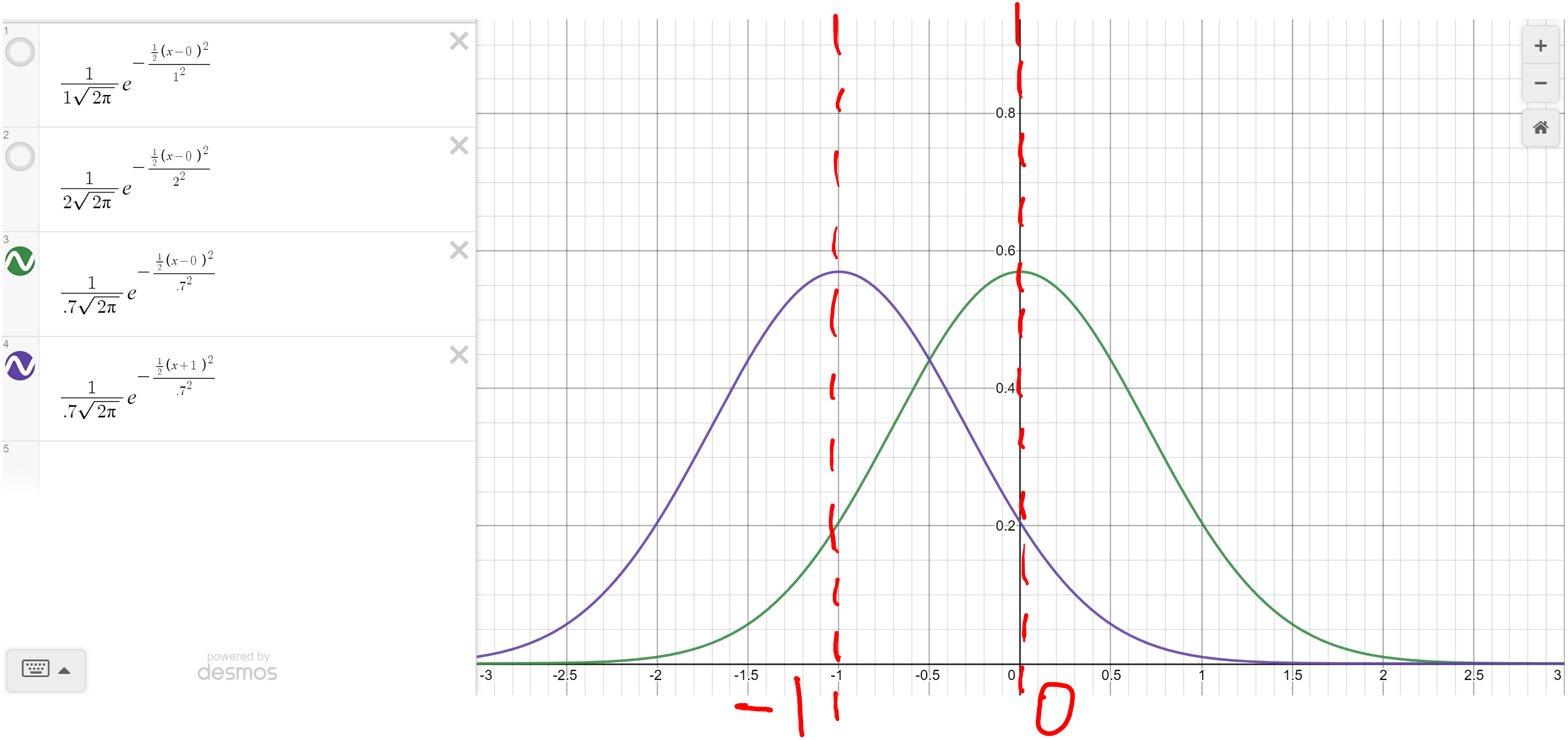
$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$



Changing the mean

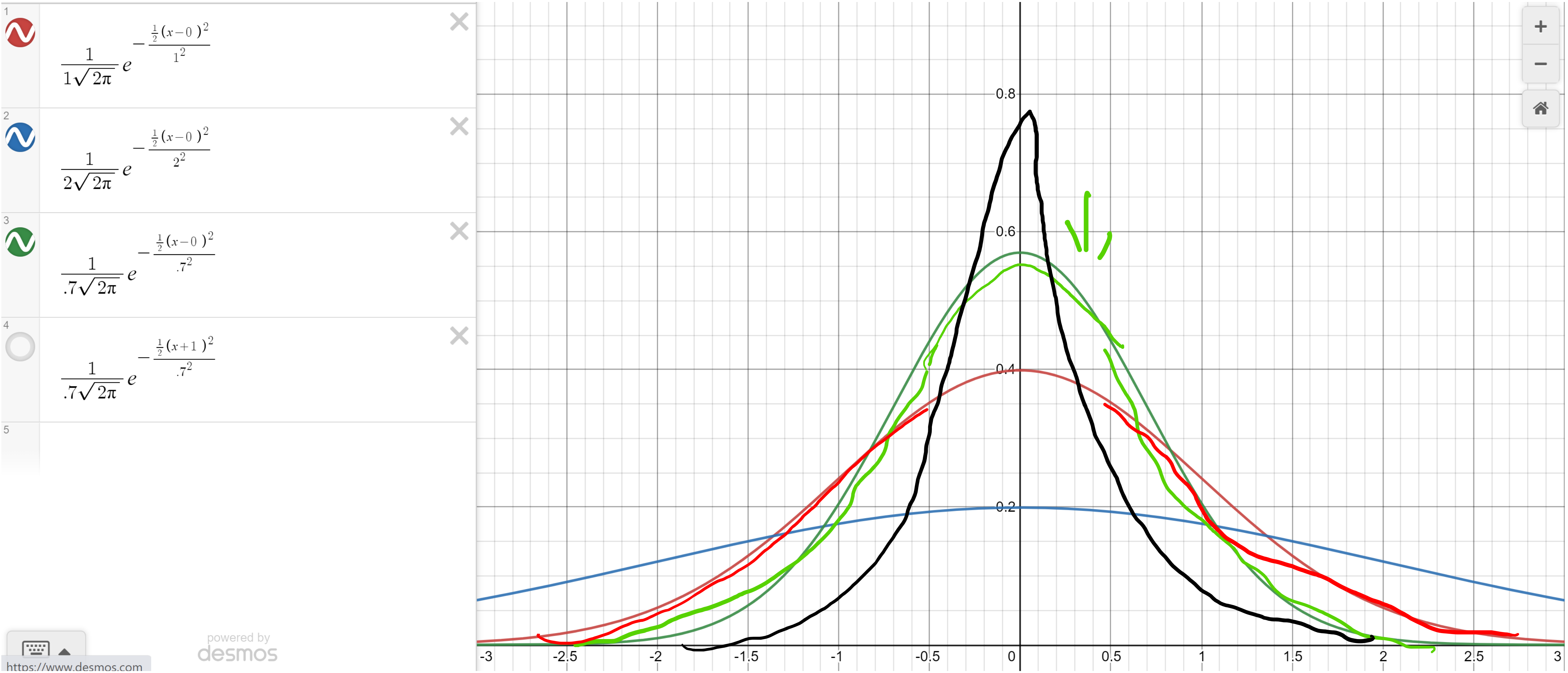
Green: $\sigma^2 = .7, \mu = 0$

Purple $\sigma^2 = .7, \mu = -1$



Changing the variance

Green: $\sigma^2 = .7$
Red $\sigma^2 = 1$
Blue: $\sigma^2 = 2$



Scaling Normal Random Variables

When we scale a normal (multiplying by a constant or adding a constant) we get a normal random variable back!

If $X \sim \mathcal{N}(\mu, \sigma^2)$

Then for $Y = aX + b$, $Y \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$

Normal are special in that you get a NORMAL back.

If you multiply a binomial by $3/2$ you don't get a binomial (its support isn't even integers!)

Standardize

$E[Y] = 0$

To turn $X \sim \mathcal{N}(\mu, \sigma^2)$ into $Y \sim \mathcal{N}(0,1)$ you want to set

$Y = \frac{X-\mu}{\sigma}$

$$E[Y] = E\left[\frac{X-\mu}{\sigma}\right] = \frac{1}{\sigma} E[X - \mu] = \frac{1}{\sigma}\left(E[X] - \mu\right)$$

Why standardize?

The density is a mess. The CDF does not have a pretty closed form.

But we're going to need the CDF a lot, so…

Table of Standard Normal CDF $Z \sim \mathcal{N}(0,1)$

The way we'll evaluate the CDF of a normal is to:

1. convert to a standard normal
2. Round the "z-score" to the hundredths place.
3. Look up the value in the table.

It's 2025, we're using a table?

The table makes sure we have consistent rounding rules (makes it easier for us to debug with you).
You can't evaluate this by hand – the "z-score" can give you intuition right away.

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.5279	0.53188	0.53586
0.1	0.53983	0.5438	0.54776	0.55172	0.55567	0.55962	0.56356	0.56749	0.57142	0.57535
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.6293	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.6591	0.66276	0.6664	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.7054	0.70884	0.71226	0.71566	0.71904	0.7224
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.7549
0.7	0.75804	0.76115	0.76424	0.7673	0.77035	0.77337	0.77637	0.77935	0.7823	0.78524
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1.0	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.8665	0.86864	0.87076	0.87286	0.87493	0.87698	0.879	0.881	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.9032	0.9049	0.90658	0.90824	0.90988	0.91149	0.91309	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.9222	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
1.6	0.9452	0.9463	0.94738	0.94845	0.9495	0.95053	0.95154	0.95254	0.95352	0.95449
1.7	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.9608	0.96164	0.96246	0.96327
1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
1.9	0.97128	0.97193	0.97257	0.9732	0.97381	0.97441	0.975	0.97558	0.97615	0.9767
2.0	0.97725	0.97778	0.97831	0.97882	0.97932	0.97982	0.9803	0.98077	0.98124	0.98169
2.1	0.98214	0.98257	0.983	0.98341	0.98382	0.98422	0.98461	0.985	0.98537	0.98574
2.2	0.9861	0.98645	0.98679	0.98713	0.98745	0.98778	0.98809	0.9884	0.9887	0.98899
2.3	0.98928	0.98956	0.98983	0.9901	0.99036	0.99061	0.99086	0.99111	0.99134	0.99158
2.4	0.9918	0.99202	0.99224	0.99245	0.99266	0.99286	0.99305	0.99324	0.99343	0.99361
2.5	0.99379	0.99396	0.99413	0.9943	0.99446	0.99461	0.99477	0.99492	0.99506	0.9952
2.6	0.99534	0.99547	0.9956	0.99573	0.99585	0.99598	0.99609	0.99621	0.99632	0.99643
2.7	0.99653	0.99664	0.99674	0.99683	0.99693	0.99702	0.99711	0.9972	0.99728	0.99736
2.8	0.99744	0.99752	0.9976	0.99767	0.99774	0.99781	0.99788	0.99795	0.99801	0.99807
2.9	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861
3.0	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.999

Use the table!

$P(X \le z)$

$Var(Y) = \sigma^2$

Standard deviation $= \sigma$

We'll use the notation $\Phi(z)$ to mean $F_X(z)$ where $X \sim \mathcal{N}(0,1)$

Let $Y \sim \mathcal{N}(5,4)$ what is $\mathbb{P}(Y > 9)$?

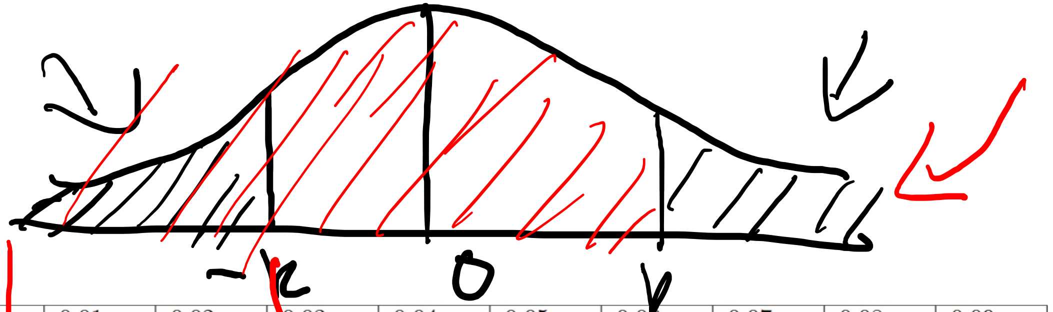
$\mathbb{P}(Y > 9)$

$= \mathbb{P}\left(\frac{Y-5}{2} > \frac{9-5}{2}\right)$ we've just written the inequality in a weird way.

$= \mathbb{P}(X > \frac{9-5}{2})$ where X is $\mathcal{N}(0,1)$.

$= 1 - \mathbb{P}\left(X \le \frac{9-5}{2}\right) = 1 - \Phi(2.00) = 1 - 0.97725 = .02275$

Negative Φ Values



$$\phi(-k) = 1 - \phi(k)$$

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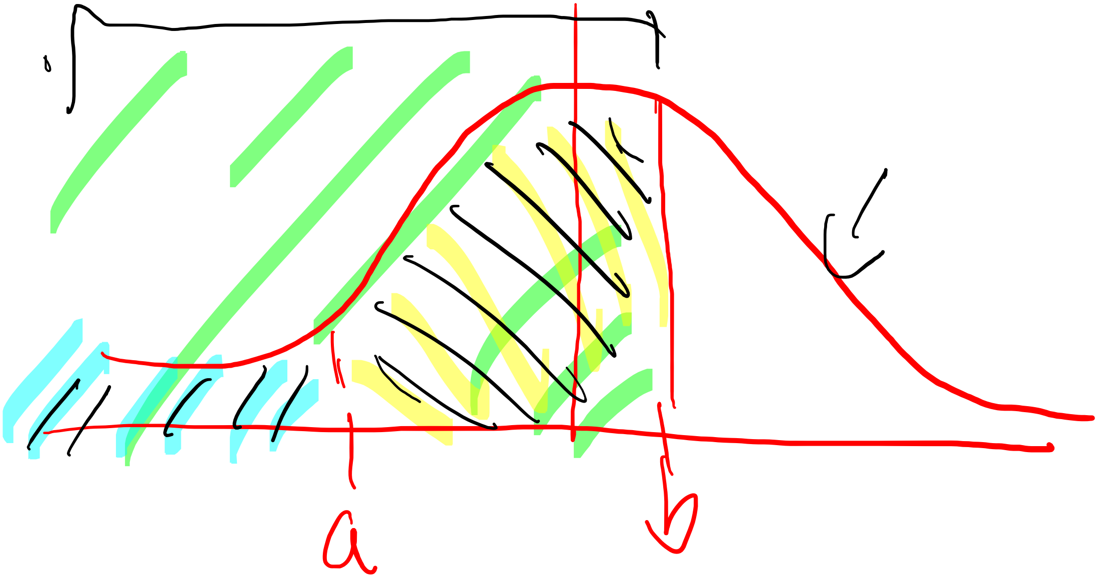
2.9	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861
3.0	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.999

$\mathbb{P}(a \leq X \leq b)$

$P(X \leq b)$



$P(X \leq a)$

$$P(a \leq X \leq b) = P(X \leq b) - P(X \leq a)$$

Normal (aka Guassian)

$X \sim \mathcal{N}(\mu, \sigma^2)$

Parameter μ is the expectation; σ^2 is the variance.

$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

$F_X(k)$ has no nice closed form. Use the table.

$\mathbb{E}[X] = \mu$

$\mathrm{Var}(X) = \sigma^2$

More practice

Let $X \sim \mathcal{N}(3, 2)$.

What is the probability that $1 \le X \le 4$

More practice

Let $X \sim \mathcal{N}(3, 2)$.

What is the probability that $1 \le X \le 4$

$$\mathbb{P}(1 \le X \le 4)$$

$$= \mathbb{P}\left(\frac{1-3}{\sqrt{2}} \le \frac{X-3}{\sqrt{2}} \le \frac{4-3}{\sqrt{2}}\right)$$

$$\approx \mathbb{P}\left(-1.41 \le \underbrace{\frac{X-3}{\sqrt{2}}} \le .71\right)$$

$$= \Phi(.71) - \Phi(-1.41)$$

$$= \Phi(.71) - \left(1 - \Phi(1.41)\right) = .76115 - (1 - .92073) = \underline{.68188}$$

In real life

What's the probability of being at most two standard deviations from the mean?

$$= \Phi(2) - \Phi(-2)$$

$$= \Phi(2) - \left(1 - \Phi(2)\right)$$

$$= .97725 - (1 - .97725) = .9545$$

You'll sometimes hear statisticians refer to the "68-95-99.7 rule" which is the probability of being within 1,2, or 3 standard deviations of the mean.

One last (unrelated) thing

A Side Note

Make sure you understand the difference between scaling a random variable and adding up iid copies of a random variable.

If X is the result of rolling a die

$X + X$ **or equivalently** $2X$ says "take the result of the (one) die roll and double it"

$2X$ has support $\{2,4,6,8,10,12\}$, there's no way to get 7 because you just double the one die roll.

If X_1, X_2 are <u>independent</u> dice rolls (i.i.d.) then

$X_1 + X_2$ says "roll two dice and add their results"

$\mathbb{E}[X_1 + X_2] = \mathbb{E}[2X]$ but $\mathrm{Var}(\mathrm{X}_1 + \mathrm{X}_2) \neq \mathrm{Var}(2X)$