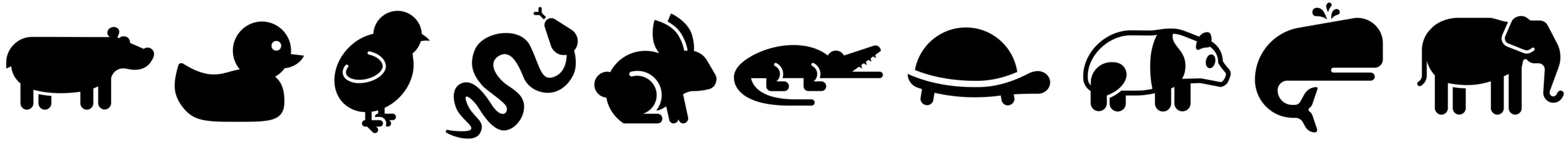
Discrete RV Zoo Part 2

CSE 312 Summer 25
Lecture 12

Zoo!



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$$\mathbb{E}[X] = \frac{a + b}{2}$$

$$\text{Var}(X) = \frac{(b - a)(b - a + 2)}{12}$$

$X \sim \text{Ber}(p)$

$$p_X(0) = 1 - p; \quad p_X(1) = p$$

$$\mathbb{E}[X] = p$$

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$X \sim \text{Bin}(n, p)$

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$$\mathbb{E}[X] = np$$

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$X \sim \text{Geo}(p)$

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$$\mathbb{E}[X] = \frac{1}{p}$$

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$X \sim \text{NegBin}(r, p)$

$$p_X(k) = \binom{k - 1}{r - 1} p^r (1 - p)^{k-r}$$

$$\mathbb{E}[X] = \frac{r}{p}$$

$$\text{Var}(X) = \frac{r(1 - p)}{p^2}$$

$X \sim \text{HypGeo}(N, K, n)$

$$p_X(k) = \frac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}}$$

$$\mathbb{E}[X] = n\frac{K}{N}$$

$$\text{Var}(X) = \frac{K(N - K)(N - n)}{N^2(N - 1)}$$

$X \sim \text{Poi}(\lambda)$

$$p_X(k) = \frac{\lambda^k e^{-\lambda}}{k!}$$

$$\mathbb{E}[X] = \lambda$$

$$\text{Var}(X) = \lambda$$

Some More Familiar Variables

Situation: Geometric

You flip a coin (which comes up heads with probability p) until you get a heads. How many flips did you need?

More generally: how many independent trials are needed until the first success?

Geometric Distribution

$X \sim \text{Geo}(p)$

p is the probability of success for one trial.

X is the number of trials needed to see the first success.

$p_X(k) = (1-p)^{k-1}p$ for $k \in \{1,2,3,\ldots\}$

$F_X(k) = 1-(1-p)^k$ for $k \in \mathbb{N}$

$\mathbb{E}[X] = \frac{1}{\mathrm{p}}$

$\text{Var}(X) = \frac{1-p}{p^2}$

Some other uses:
How many bits can we write before one is incorrect?
How many questions do you have to answer until you get one right?
How many times can you run an experiment until it fails for the first time?

Geometric: Expectation

$\mathbb{E}[X] = \sum_{k=1}^{\infty} k(1-p)^{k-1}p$

$= p\sum_{k=1}^{\infty} k(1-p)^{k-1} = p \cdot \frac{1}{p^2} = \frac{1}{p}.$

Intuition: Smaller p means longer wait

$\text{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$

$= \frac{2-p}{p^2} - \frac{1}{p^2} = \frac{1-p}{p^2}$

Intuition: for small p lots of variance (might have to wait a long time, and it's variable)
For large p very little variance (for $p = 1$ there's no variation at all!)

Geometric Property

Geometric random variables are called “memoryless”

Suppose you’re flipping coins (independently) until you see a heads.

The first two came up tails.

How many flips are ***left*** until you see the first heads?

It’s another independent copy of the original!

The coin “forgot” it already came up tails 2 times.

Formally…

Let X be the total number of flips needed, Y be the flips after the second.

$\mathbb{P}(Y = k | X \geq 3) =?$

…

Which is $p_X(k)$.

Formally…

Let X be the total number of flips needed, Y be the flips after the second.

$\mathbb{P}(Y = k | X \geq 3) = \mathbb{P}(Y = k \cap X \geq 3)/\mathbb{P}(X \geq 3)$

$\frac{(1-p)^{k+2-1}p}{(1-p)^2}$

$= (1-p)^{k-1}p$

Which is $p_X(k)$.

So, the (conditional) PMF for Y matches that of X. The coin "forgot" it did the first two flips.

Scenario: Uniform

You Roll a Fair Die (or draw a random integer from 1,…,n).

More generally: you want an integer in some range, with each equally likely.

Discrete Uniform Distribution

$X \sim \text{Unif}(a, b)$

Parameter a is the minimum value in the support, b is the maximum value in the support.

X is a uniformly random integer between a and b (inclusive)

$p_X(k) = \frac{1}{b-a+1}$ for $k \in \mathbb{Z}$, $a \leq k \leq b$

$F_X(k) = \frac{k-a+1}{b-a+1}$ for $k \in \mathbb{Z}$, $a \leq k \leq b$.

$\mathbb{E}[X] = \frac{a+b}{2}$

$\text{Var}(X) = \frac{(b-a)(b-a+2)}{12}$

Some Less Familiar Distributions

The Poisson Distribution

A new kind of random variable.

We use a Poisson distribution when:

We're trying to count the number of times something happens in some interval of time.

We know the average number that happen (i.e. the expectation)

Each occurrence is independent of the others.

There are a VERY large number of "potential sources" for those events, few of which happen.

The Poisson Distribution

Classic applications:

- How many traffic accidents occur in Seattle in a day?
- How many major earthquakes occur in a year (not including aftershocks)?
- How many customers visit a bakery in an hour?

Why not just use counting coin flips?

What are the flips…the number of cars? Every person who might visit the bakery? There are way too many of these to count exactly or think about dependency between.

But a Poisson might accurately model what's happening.

It’s a model

By modeling choice, we mean that we’re choosing math that we think represents the real world as best as possible

Is every traffic accident really independent?

Not *really*, one causes congestion, which causes angrier drivers. Or both might be caused by bad weather/more cars on the road.

But we assume they are (because the dependence is so weak that the model is useful).

Poisson Distribution

$X \sim \text{Poi}(\lambda)$

Let λ be the average number of incidents in a time interval.

X is the number of incidents seen in a particular interval.

Support $\mathbb{N}$

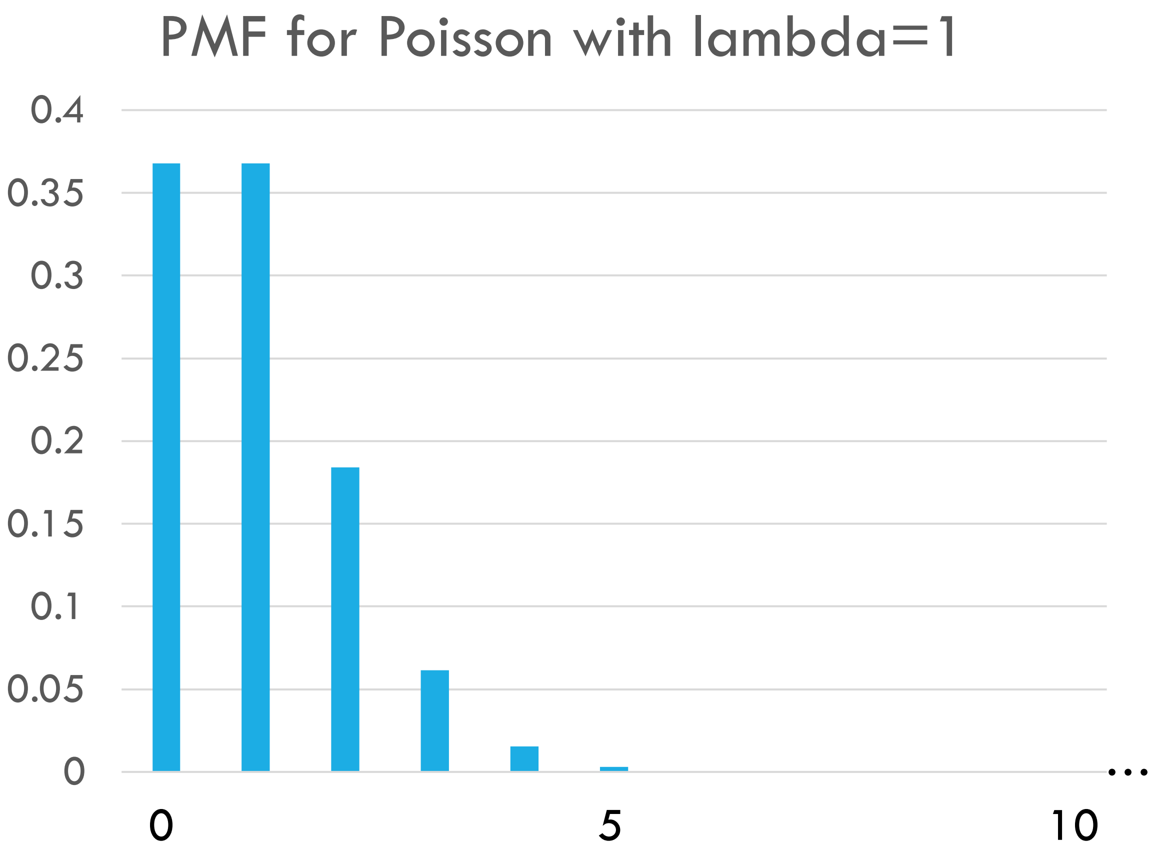
$p_X(k) = \frac{\lambda^k e^{-\lambda}}{k!}$ (for $k \in \mathbb{N}$)

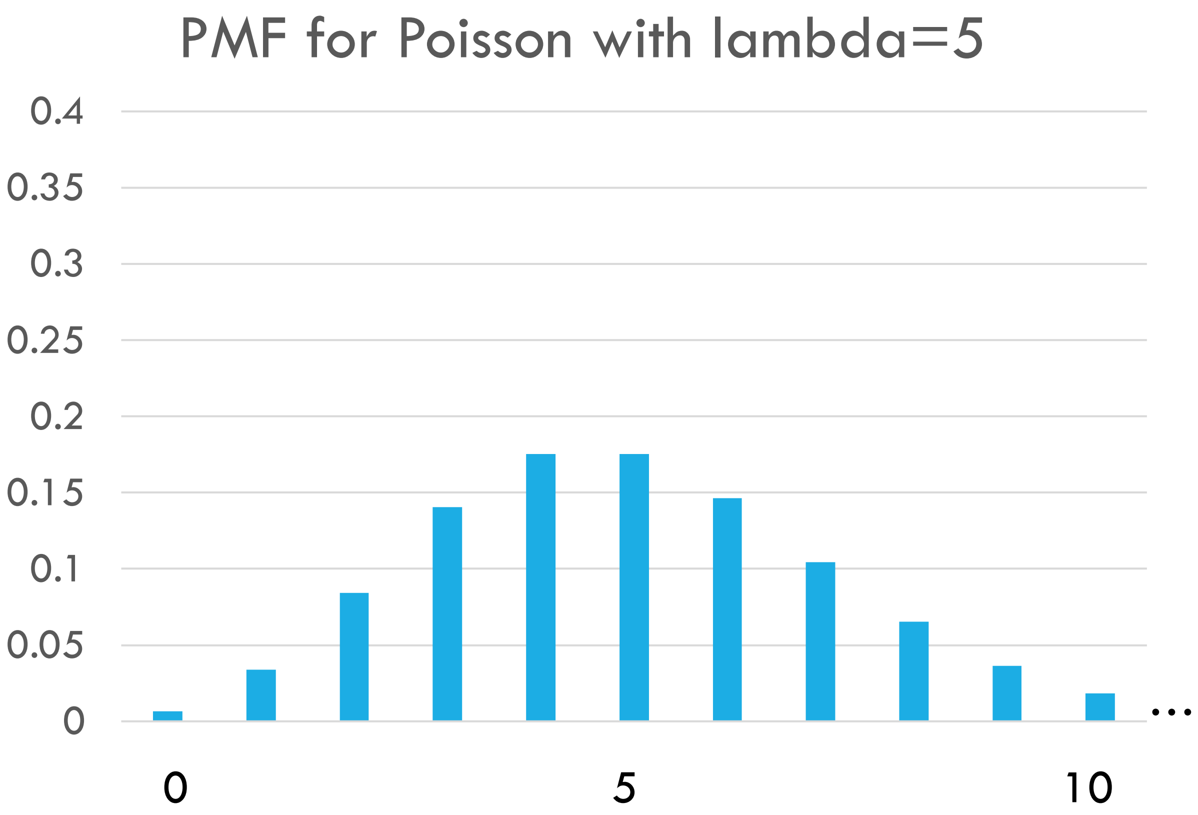
$F_X(k) = e^{-\lambda} \sum_{i=0}^{\lfloor k \rfloor} \frac{\lambda^i}{i!}$

$\mathbb{E}[X] = \lambda$

$\text{Var}(X) = \lambda$

Some Sample PMFs





Let's take a closer look at that pmf

$p_X(k) = \frac{\lambda^k e^{-\lambda}}{k!}$ (for $k \in \mathbb{N}$)

If this is a real PMF, it should sum to 1.
Let's check that to understand the PMF a little better.

$\sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!}$

$= e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}$

Taylor Series for e^x

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x$$

$= e^{-\lambda} e^{\lambda} = e^0 = 1$

Let's check something…the expectation

$\mathbb{E}[X] = \sum_{k=0}^{\infty} k \cdot e^{-\lambda} \frac{\lambda^k}{k!}$

$= \sum_{k=1}^{\infty} k \cdot e^{-\lambda} \frac{\lambda^k}{k!}$ first term is 0.

$= \sum_{k=1}^{\infty} e^{-\lambda} \frac{\lambda^k}{(k-1)!}$ cancel the k.

$= \lambda \sum_{k=1}^{\infty} e^{-\lambda} \frac{\lambda^{k-1}}{(k-1)!}$ factor out λ.

$= \lambda \sum_{j=0}^{\infty} e^{-\lambda} \frac{\lambda^j}{(j)!}$ Define $j = k - 1$

$= \lambda \cdot 1$ The summation is just the pmf!

Where did this expression come from?

For the cars we said "it's like every car in Seattle independently might cause an accident."

If we knew the exact number of cars, and they all had identical probabilities of causing an accident…

It'd be just like counting the number of heads in n flips of a bunch of coins (the coins are just VERY biased).

The Poisson is a certain limit as $n \to \infty$ but np (the expected number of accidents) stays constant.

Scenario: Negative Binomial

You're playing a carnival game, and there are r little kids nearby who all want a stuffed animal. You can win a single game (and thus win one stuffed animal) with probability p (independently each time) How many times will you need to play the game before every kid gets their toy?

More generally, run independent trials with probability p. How many trials do you need for r successes?

Try it

More generally, run independent trials with probability p. How many trials do you need for r successes?

What's the pmf?

What's the expectation and variance (hint: linearity)

Negative Binomial Analysis

What's the pmf? Well how would we know $X = k$?

Of the first $k-1$ trials, $r-1$ must be successes.
And trial k must be a success.

That first part is a lot like a binomial!

It's the $p_Y(r-1)$ where $Y \sim \text{Bin}(k-1, p)$

First part gives $\binom{k-1}{r-1}(1-p)^{k-1-(r-1)}p^{r-1} = \binom{k-1}{r-1}(1-p)^{k-r}p^{r-1}$

Second part, multiply by p

Total: $p_X(k) = \binom{k-1}{r-1}(1-p)^{k-r}p^r$

Negative Binomial Analysis

What about the expectation?

To see r successes:

We flip until we see success 1.

Then flip until success 2.

... Flip until success r.

The total number of flips is...the sum of geometric random variables!

Negative Binomial Analysis

Let $Z_1, Z_2, \dots, Z_r$ be independent copies of $\text{Geo}(p)$

Z_i are called "independent and identically distributed" or "i.i.d.'

Because they are independent…and have identical pmfs.

$X \sim \text{NegBin}(r, p)$ $X = Z_1 + Z_2 + \cdots + Z_r$.

$\mathbb{E}[X] = \mathbb{E}[Z_1 + Z_2 + \cdots Z_r] = \mathbb{E}[Z_1] + \mathbb{E}[Z_2] + \cdots + \mathbb{E}[Z_r] = r \cdot \frac{1}{p}$

Negative Binomial Analysis

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$X \sim \text{NegBin}(r, p)$ $X = Z_1 + Z_2 + \cdots + Z_r$.

$\text{Var}(X) = \text{Var}(Z_1 + Z_2 + \cdots + Z_r)$

Up until now we've just used the observation that $X = Z_1 + \cdots + Z_r$.

$= \text{Var}(Z_1) + \text{Var}(Z_2) + \cdots + \text{Var}(Z_r)$ because the Z_i are independent.

$= r \cdot \frac{1-p}{p^2}$

Negative Binomial

$X \sim \mathrm{NegBin}(\mathrm{r}, \mathrm{p})$

Parameters: r: the number of successes needed, p the probability of success in a single trial

X is the number of trials needed to get the r^{th} success.

$p_X(k) = \binom{k-1}{r-1}(1-p)^{k-r}p^r$

$F_X(k)$ is ugly, don't bother with it.

$\mathbb{E}[X] = \frac{r}{p}$

$\mathrm{Var}(\mathrm{X}) = \frac{\mathrm{r}(1-\mathrm{p})}{p^2}$

Scenario: Hypergeometric

You have an urn with N balls, of which K are purple. You are going to draw balls out of the urn **without** replacement.

If you draw out n balls, what is the probability you see k purple ones?

Hypergeometric: Analysis

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If you draw out n balls, what is the probability you see k purple ones?

Of the K purple, we draw out k, choose which k will be drawn

Of the $N - K$ other balls, we will draw out $n - k$, choose which $(n - k)$ will be removed.

Sample space all subsets of size n from N balls

$$\frac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}}$$

Hypergeometric: Analysis

$X = D_1 + D_2 + \cdots + D_n$

Where D_i is the indicator that draw i is purple.

D_1 is 1 with probability K/N.

What about D_2?

$\mathbb{P}(D_2 = 1) = \frac{K-1}{N-1} \cdot \frac{K}{N} + \frac{K}{N-1} \cdot \frac{K-N}{N} = \frac{K(K-N+K-1)}{N(N-1)} = \frac{K}{N}$

In general $\mathbb{P}(D_i = 1) = \frac{K}{N}$

It might feel counterintuitive, but it's true!

Hypergeometric: Analysis

$\mathbb{E}[X]$

$= \mathbb{E}[D_1 + \cdots D_n] = \mathbb{E}[D_1] + \cdots + \mathbb{E}[D_n] = n \cdot \frac{K}{N}$

Can we do the same for variance?

No! The D_i are dependent. Even if they have the same probability.

Hypergeometric Random Variable

$X \sim \text{HypGeo}(N, K, n)$

Parameters: A total of N balls in an urn, of which K are successes. Draw n balls without replacement.

X is the number of success balls drawn.

$$p_X(k) = \frac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}}$$

$$\mathbb{E}[X] = \frac{nK}{N}$$

$$\text{Var}(X) = n \cdot \frac{K}{N} \cdot \frac{N-K}{N} \cdot \frac{N-n}{N-1}$$

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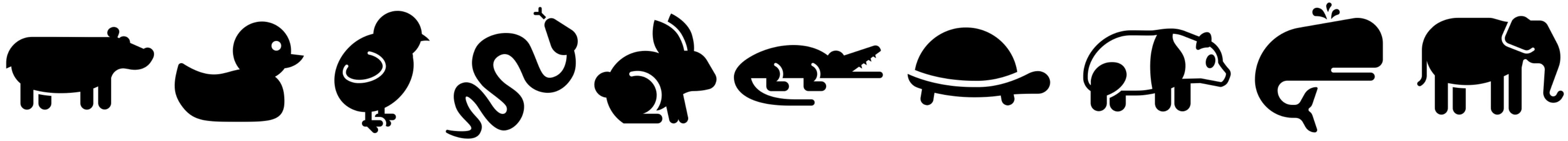

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Zoo Takeaways

You can do relatively complicated counting/probability calculations much more quickly than you could week 1!

You can now explain why your problem is a zoo variable and save explanation on homework (and save yourself calculations in the future).

Don’t spend extra effort memorizing…but be careful when looking up Wikipedia articles.

The exact definitions of the parameters can differ (is a geometric random variable the number of failures before the first success, or the total number of trials including the success?)

Practice Problem: Poisson

Seattle averages 3 days with snowfall per year.

Suppose that the number of days with snow follows a Poisson distribution. What is the probability of getting exactly 5 days of snow?

According to the Poisson model, what is the probability of getting 367 days of snow?

Practice: Poisson

Let $X \sim \mathrm{Poi}(3)$.

$f_X(5) = \frac{3^5 e^{-3}}{5!} \approx .1008$

Or about once a decade.

Probability of 367 snowy days, err…

The distribution says

$f_X(367) \approx 1.8\text{x}10^{-610}$.

Definition of a "year" says probability should be 0.