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- outside Allen 206

Midterm Prep
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- practice
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Expectation and Variance

CSE 312 Summer 25
Lecture 9

Random Variable

What's a random variable?

Formally

Random Variable

$X: \Omega \to \mathbb{R}$ is a random variable
$X(\omega)$ is the summary of the outcome ω

Informally: A random variable is a way to **summarize** the important (numerical) information from your outcome.

Support Ω_X
the set of values X can take.

Probability Mass Function (pmf $p_X(x)$) on input x, tells you $\mathbb{P}(X = x)$.

Try It Yourself

There are 20 balls, numbered 1,2,…,20 in a jar.

You'll draw out a size-three subset. (i.e. without replacement)

$\Omega = \{$size three subsets of $\{1, \ldots, 20\}$ $\}$, $\mathbb{P}()$ is uniform measure.

Let X be the largest value among the three balls.

If outcome is $\{4,2,10\}$ then $X = 10$.

Write down the pmf of X

Try It Yourself

$X = 7$

$$\frac{\binom{6}{2}}{\binom{20}{3}}$$

There are 20 balls, numbered 1,2,...,20 in a jar.

You'll draw out a size-three subset. (i.e. without replacement)

Let X be the largest value among the three balls.

$\{\_\ \_\ 7\}$

$$p_X(x) = \begin{cases} \binom{x-1}{2} / \binom{20}{3} & \text{if } x \in \mathbb{N}, 3 \leq x \leq 20 \\ 0 & \text{otherwise} \end{cases}$$

$\binom{6}{2}$

Good check: if you sum up $p_X(x)$ do you get 1?

$P(X = 7)$

Good check: is $p_X(x) \geq 0$ for all x? Is it defined for all x?

Describing a Random Variable

The most common way to describe a random variable is the PMF.

But there's a second representation:

The cumulative distribution function (CDF) gives the probability $X \leq x$

More formally, $\mathbb{P}(\{\omega : X(\omega) \leq x\})$

Often written $F_X(x) = \mathbb{P}(X \leq x)$

$F_X(x) = \sum_{i: i \leq x} p_X(i)$

Try it yourself

What is the CDF of X where

X be the largest value among the three balls. (Drawing 3 of the 20 without replacement)

$P(X \leq x)$

$\binom{7}{3}$

$\rightarrow P(X \leq 7)$

$\{1, 2, 6\}$

$P(X \leq 7.5)$

$\sum p_X(x)$

$$F_X(x) = \begin{cases} 0 & x < 3 \\ \binom{\lfloor x \rfloor}{3} \Big/ \binom{20}{3} & 3 \leq x \leq 20 \\ 1 & \text{otherwise} \end{cases}$$

Try it yourself

What is the CDF of X where

X be the largest value among the three balls. (Drawing 3 of the 20 without replacement)

$$F_X(x) = \begin{cases} 0 & \text{if } x < 3 \\ \binom{\lfloor x \rfloor}{3} / \binom{20}{3} & \text{if } 3 \le x \le 20 \\ 1 & \text{otherwise} \end{cases}$$

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What is the CDF of X where

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$$F_X(x) = \begin{cases} 0 & \text{if } x < 3 \\ \binom{\lfloor x \rfloor}{3} / \binom{20}{3} & \text{if } 3 \leq x \leq 20 \\ 1 & \text{otherwise} \end{cases}$$

Good checks: Is $F_X(\infty) = 1$?

Is $F_X(x)$ increasing?

Is $F_X(x)$ defined for all real number inputs?

Two descriptions

$P(X=x)$

$P(X\leq x)$

PROBABILITY MASS FUNCTION

Defined for all $\mathbb{R}$ inputs.

Usually has "0 otherwise" as an extra case.

$\sum_x p_X(x) = 1$

$0 \leq p_X(x) \leq 1$

$\sum_{z:z\leq x} p_X(z) = F_X(x)$

CUMULATIVE DISTRIBUTION FUNCTION

Defined for all $\mathbb{R}$ inputs.

Often has 0 and 1 extra cases

Non-decreasing function

$0 \leq F_X(x) \leq 1$

$\lim_{x\to-\infty} F_X(x) = 0$

$\lim_{x\to\infty} F_X(x) = 1$

More Independence

Independence of Random Variables

Independence (of random variables) Ω_X Ω_Y

X and Y are independent if for all k, ℓ

$$\mathbb{P}(X = k, Y = \ell) = \mathbb{P}(X = k)\mathbb{P}(Y = \ell)$$

We'll often use commas instead of ∩ symbol.

Independence of Random Variables

The “for all values” is important.

We say that the event “the sum is 7” is independent of “the red die is 5”

What about S =“the sum of two dice” and R =“the value of the red die”

Independence of Random Variables

The "for all values" is important.

We say that the event "the sum is 7" is independent of "the red die is 5"

What about S ="the sum of two dice" and R ="the value of the red die"

NOT independent.

$$\underbrace{\mathbb{P}(S=2, R=5)}_{1/36} \neq \underbrace{\mathbb{P}(S=2)\mathbb{P}(R=5)}_{\frac{1}{6}}$$ (for example)

Independence of Random Variables

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Flip a coin independently $2n$ times.

Let X be "the number of heads in the first n flips."

Let Y be "the number of heads in the last n flips."

$2n-1$

$2, 2n$

X and Y are independent.

$$P(X=2, Y=3)$$
$$= P(X=2) \cdot P(Y=3)$$

Mutual Independence for RVs

A little simpler to write down than for events

Mutual Independence (of random variables)

$X_1, X_2, \dots, X_n$ **are mutually independent if for all** $x_1, x_2, \dots, x_n$

$$\mathbb{P}(X_1 = x_1, X_2 = x_2, \dots, X_n = x_n) = \mathbb{P}(X_1 = x_1)\mathbb{P}(X_2 = x_2) \cdots \mathbb{P}(X_n = x_n)$$

DON'T need to check all subsets for random variables…

But you do need to check all values (all possible x_i) still.

Expectation

Expectation

$\mathbb{E}$

Expectation

The "expectation" (or "expected value") of a random variable X is:

$$\mathbb{E}[X] = \sum_{k \in \Omega_X} k \cdot \mathbb{P}(X = k)$$

Intuition: The weighted average of values X could take on.

Weighted by the probability you actually see them.

Example 1

Flip a fair coin twice (independently)

Let X be the number of heads.

$\Omega = \{TT, TH, HT, HH\}$, $\mathbb{P}()$ is uniform measure.

$$\mathbb{E}[X] = \frac{1}{4} \cdot 0 + \frac{1}{2} \cdot 1 + \frac{1}{4} \cdot 2 = 0 + \frac{1}{2} + \frac{1}{2} = 1.$$

Example 2

$$\sum_{k \in \Omega_X} k \cdot P(X=k)$$

You roll a biased die.

It shows a 6 with probability $\frac{1}{3}$, and 1,…,5 with probability $2/15$ each.

Let X be the value of the die. What is $\mathbb{E}[X]$?

$$\frac{1}{3} \cdot 6 + \frac{2}{15} \cdot 5 + \frac{2}{15} \cdot 4 + \frac{2}{15} \cdot 3 + \frac{2}{15} \cdot 2 + \frac{2}{15} \cdot 1$$

$$= 2 + \frac{2(5+4+3+2+1)}{15} = 2 + \frac{30}{15} = 4$$

$\mathbb{E}[X]$ is not just the most likely outcome!

Try it yourself

Let X be the result shown on a fair die. What is $\mathbb{E}[X]$? $\mathbb{E}[X] = 3.5$

Let Y be the sum of two (independent) fair die rolls. What is $\mathbb{E}[Y]$?

Try it yourself

Let X be the result shown on a fair die. What is $\mathbb{E}[X]$

$$6 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 1 \cdot \frac{1}{6}$$

$$= \frac{21}{6} = 3.5$$

$\mathbb{E}[X]$ is not necessarily a possible outcome!

That's ok, it's an average!

Try it yourself

Let Y be the sum of two (independent) fair die rolls. What is $\mathbb{E}[Y]$?

$\mathbb{E}[Y] =$

$$\frac{1}{36} \cdot 2 + \frac{2}{36} \cdot 3 + \frac{3}{36} \cdot 4 + \frac{4}{36} \cdot 5 + \frac{5}{36} \cdot 6 + \frac{6}{36} \cdot 7 + \frac{5}{36} \cdot 8 + \frac{4}{36} \cdot 9 + \frac{3}{36} \cdot 10 + \frac{2}{36} \cdot 11 + \frac{1}{36} \cdot 12$$

$= 7$

$\mathbb{E}[Y] = 2\mathbb{E}[X]$. That's not a coincidence…we'll talk about why on Wednesday.

Subtle but Important

X is random. You don't know what it is (at least until you run the experiment).

$\mathbb{E}[X]$ is not random. It's a number.
You don't need to run the experiment to know what it is.

Functions of a random variable

Let X, Y be random variables defined on the same sample space.

Functions of X and/or Y like

$$X + Y$$
$$X^2$$
$$2X + 3$$

c

2x+3 6 15

Are random variables! (Say what the outcome is, and these functions give you a number. They're functions from $\Omega \to \mathbb{R}$. **That's the definition of a random variable!**

Expectations of functions of random variables

$E[X] = \sum_{k \in \Omega_X} k \cdot P(X=k)$

$X \leftarrow \omega$

Let's say we have a random variable X and a function g. What is $\mathbb{E}[g(X)]$?

$g(X)$

$$\mathbb{E}[g(X)] = \sum_{k \in \Omega_{g(X)}} k \cdot \mathbb{P}(g(X) = k)$$

$\leftarrow P_{g(x)}(x)$

X, Y are ind.

$E[X \cdot Y] = E[X] \cdot E[Y]$

$$\mathbb{E}[g(X)] = \sum_{\omega \in \Omega} g(X(\omega)) \cdot \mathbb{P}(\omega)$$

$$\mathbb{E}[g(X)] = \sum_{k \in \Omega_X} g(k) \cdot \mathbb{P}(X = k)$$

Law of the Unconscious Statistician (LOTUS)

$X \quad X^2$

Notice that $\mathbb{E}[g(X)]$ might not be $g(\mathbb{E}[X])$.

$E[X^2] \quad (E[X])^2$

Variance

Where are we?

A random variable is a way to summarize what outcome you saw.

The Expectation of a random variable is its average value.

A way to summarize a random variable

Variance

Another one number summary of a random variable.

But wait, we already have expectation, what's this for?

Consider these two games

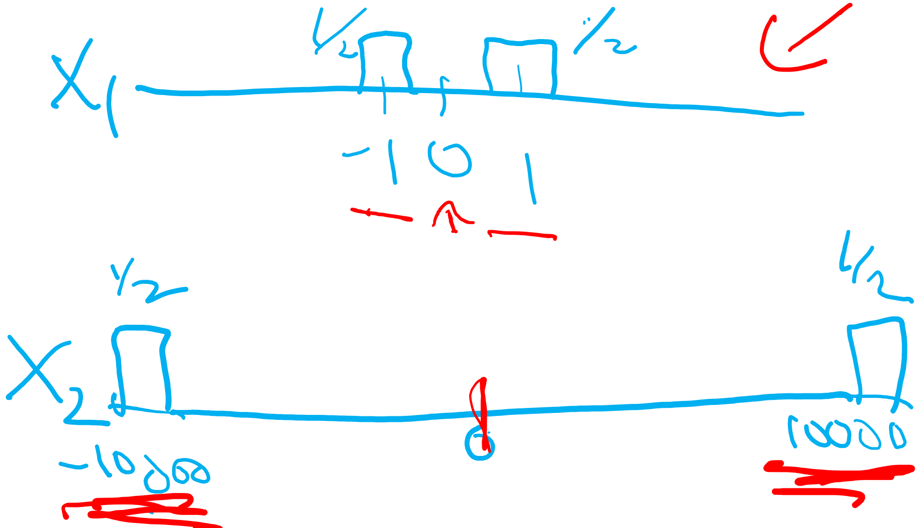
Would you be willing to play these games?

Game 1: I will flip a fair coin; if it's heads, I pay you \$1. If it's tails, you pay me \$1. Let X_1 be your profit if you play game 1

Game 2: I will flip a fair coin; if it's heads, I pay you \$10,000. If it's tails, you pay me \$10,000. Let X_2 be your profit if you play game 2.

Both games are "fair" ($\mathbb{E}[X_1] = \mathbb{E}[X_2] = 0$)

Consider these two games



Would you be willing to play these games?

Game 1: I will flip a fair coin; if it's heads, I pay you \$1. If it's tails, you pay me \$1. Let X_1 be your profit if you play game 1

Game 2: I will flip a fair coin; if it's heads, I pay you \$10,000. If it's tails, you pay me \$10,000. Let X_2 be your profit if you play game 2.

Both games are "fair" ($\mathbb{E}[X_1] = \mathbb{E}[X_2] = 0$)

What’s the difference

Expectation tells you what the average will be…

But it doesn’t tell you how “extreme” your results could be.

Nor how likely those extreme results are.

Game 2 has many (well, only) very extreme results.

In expectation they “cancel out” but if you can only play once…

…it would be nice to measure that.

Variance

Variance

The variance of a random variable X is

$$\mathrm{Var}(X) = \sum_{\omega} \mathbb{P}(\omega) \cdot (X(\omega) - \mathbb{E}[X])^2 = \mathbb{E}[(X - \mathbb{E}[X])^2] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

$E[X^2]$

The first two forms are the definition. The last one is an algebra trick.

$E[X]^2$

$E[X^2]$ $E[X]^2$

Proof of Calculation Trick

$\mathbb{E}[(X - \mathbb{E}[X])^2] = \mathbb{E}[X^2 - 2X\mathbb{E}[X] + (\mathbb{E}[X])^2]$ expanding the square

$= \mathbb{E}[X^2] - \mathbb{E}\big[2X\mathbb{E}[X]\big] + \mathbb{E}[(\mathbb{E}[X])^2]$ linearity of expectation.

$= \mathbb{E}[X^2] - 2\mathbb{E}[X]\mathbb{E}[X] + \mathbb{E}[(\mathbb{E}[X])^2]$ linearity of expectation.

$= \mathbb{E}[X^2] - 2\mathbb{E}[X]\mathbb{E}[X] + (\mathbb{E}[X])^2$ expectation of a constant is the constant

$= \mathbb{E}[X^2] - 2(\mathbb{E}[X])^2 + (\mathbb{E}[X])^2$

$= \mathbb{E}[X^2] - (\mathbb{E}[X])^2$

So $\mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$.

Variance of a die

Let X be the result of rolling a fair die.

$\mathrm{Var}(\mathrm{X}) = \mathbb{E}[(X - \mathbb{E}[X])^2] = \mathbb{E}[(X - 3.5)^2]$

$= \frac{1}{6}(1 - 3.5)^2 + \frac{1}{6}(2 - 3.5)^2 + \frac{1}{6}(3 - 3.5)^2 + \frac{1}{6}(4 - 3.5)^2 + \frac{1}{6}(5 - 3.5)^2 + \frac{1}{6}(6 - 3.5)^2$

$= \frac{35}{12} \approx 2.92.$

Or $\mathbb{E}[X^2] - (E[X])^2 = \sum_{k=1}^{6} \frac{1}{6} \cdot k^2 - 3.5^2 = \frac{91}{6} - 3.5^2 \approx 2.92$

Variance

If X and Y are independent then
$$\mathbf{Var}(X+Y)=\mathbf{Var}(X)+\mathbf{Var}(Y)$$

Let X be the result of rolling a fair die.

Let Y be the result of rolling a fair die.

$$\mathrm{Var}(\mathrm{X}+Y) = \mathrm{Var}(\mathrm{X}) + \mathrm{Var}(\mathrm{Y}) \approx 5.84$$

Useful Facts

Facts About Variance

$$\mathrm{Var}(X + c) = \mathrm{Var}(X)$$

Proof:

$$\mathrm{Var}(X + c) = \mathbb{E}[(X + c)^2] - \mathbb{E}[X + c]^2$$

$$= \mathbb{E}[X^2] + \mathbb{E}[2Xc] + \mathbb{E}[c^2] - (\mathbb{E}[X] + c)^2$$

$$= \mathbb{E}[X^2] + 2c\mathbb{E}[X] + c^2 - \mathbb{E}[X]^2 - 2c\mathbb{E}[X] - c^2$$

$$= \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

$$= \mathrm{Var}(X)$$

Facts about Variance

$$\mathrm{Var}(aX) = a^2\mathrm{Var}(X)$$

$$= \mathbb{E}[(aX)^2] - (\mathbb{E}[aX])^2$$

$$= a^2\mathbb{E}[X^2] - (a\mathbb{E}[X])^2$$

$$= a^2\mathbb{E}[X^2] - a^2\mathbb{E}[X]^2$$

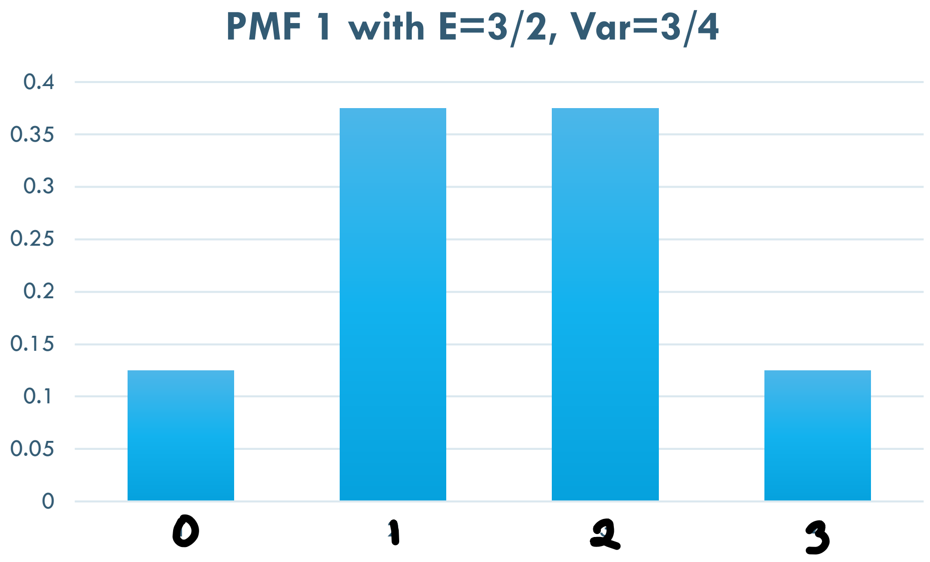
$$= a^2(\mathbb{E}[X^2] - \mathbb{E}[X]^2)$$

Expectation and Variance aren’t everything

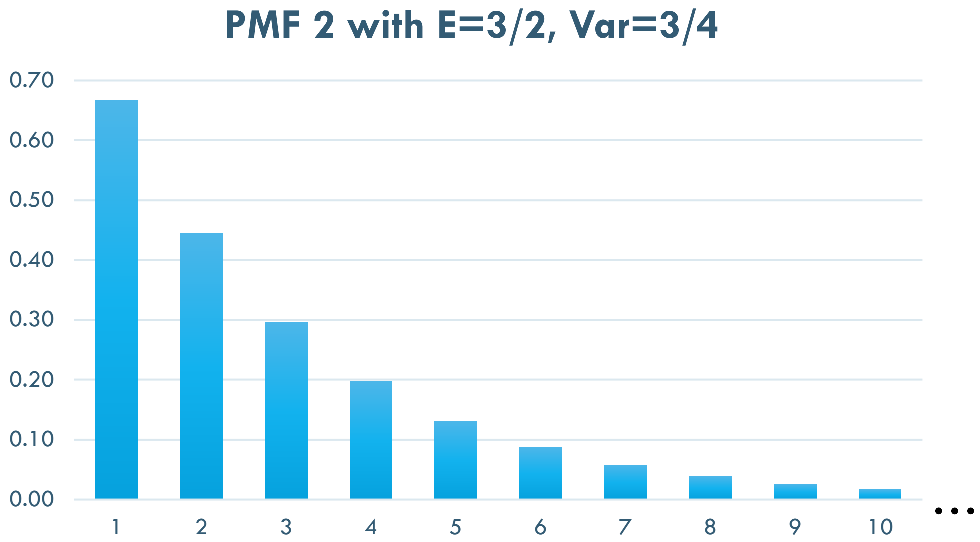
Alright, so expectation and variance is everything right?

No!

Flip a fair coin 3 times indep. Count heads.



Flip a biased coin (prob heads=2/3) until heads. Count flips.



A PMF or CDF *does* fully describe a random variable.

Extra Practice

More Practice

Suppose you flip a coin until you see a heads for the first time.

Let X be the number of trials (including the heads)

What is the pmf of X?

The cdf of X?

$\mathbb{E}[X]$?

More Practice

Suppose you flip a coin until you see a heads for the first time.

Let X be the number of trials (including the heads)

What is the pmf of X? $f_X(x) = 1/2^x$ for $x \in \mathbb{Z}^+$, 0 otherwise

The cdf of X? $F_X(x) = 1 - 1/2^{\lfloor x \rfloor}$ for $x \geq 0$, 0 for $x < 0$.

$\mathbb{E}[X]$? $\sum_{i=1}^{\infty} \frac{i}{2^i} = 2$

More Random Variable Practice

Roll a fair die n times. Let X be the number of rolls that are $5s$ or $6s$.

What is the pmf?

Don’t try to write the CDF…it’s a mess…

Or try for a few minutes to realize it isn’t nice.

What is the expectation?

More Random Variable Practice

Roll a fair die n times. Let Z be the number of rolls that are $5s$ or $6s$.

What's the probability of getting exactly k 5's/6's? Well we need to know which k of the n rolls are 5's/6's. And then multiply by the probability of getting exactly that outcome

$$f_Z(z) = \begin{cases} \binom{n}{z} \cdot \left(\frac{1}{3}\right)^z \left(\frac{2}{3}\right)^{n-z} & \text{if } z \in Z, 0 \leq z \leq n \\ 0 & \text{otherwise} \end{cases}$$

Expectation formula is a mess. If you plug it into a calculator you'll get a nice, clean simplification: $n/3$.