Independence

CSE 312 Summer 25
Lecture 7

Announcements

HW2 due tonight

Penalty free late days require a meaningful attempt by the original deadline.

HW3 out this evening.

HW3 includes a programming problem – using Bayes rule to do some machine learning – detecting whether emails are spam or “ham” (legitimate emails).

Longer than the programming on HW1 – please get started early!

Extra resources will be available!

The Technical Stuff

Proof of Bayes' Rule

$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$ by definition of conditional probability

Now, imagining we get $A \cap B$ by conditioning on A, we should get a numerator of $\mathbb{P}(B|A) \cdot \mathbb{P}(A)$

$= \frac{\mathbb{P}(B|A) \cdot \mathbb{P}(A)}{\mathbb{P}(B)}$

As required.

A Quick Technical Remark

Sometimes students write things like

$\mathbb{P}([A|B]|C)$

This is not a thing.

You probably want $\mathbb{P}(A|[B \cap C])$

$A|B$ isn't an event – it's describing an event **and** telling you to restrict the sample space. So you can't ask for the probability of that conditioned on something else.

A Technical Note

After you condition on an event, what remains is a probability space.

With B playing the role of the sample space,

$\mathbb{P}(\omega|B)$ playing the role of the probability measure.

All the axioms are satisfied (it's a good exercise to check)

That means any theorem we write down has a version where you condition everything on B.

An Example

Bayes Theorem still works in a probability space where we've already conditioned on S.

$$\mathbb{P}(A|[B \cap S]) = \frac{\mathbb{P}(B|[A \cap S]) \cdot \mathbb{P}(A|S)}{\mathbb{P}(B|S)}$$

Complementary law still works in a probability space where we've already conditioned on S

$$\mathbb{P}(A|C) = 1 - \mathbb{P}(\bar{A}|C)$$

Independence

Definition of Independence

We've calculated conditional probabilities.

Sometimes conditioning – getting some partial information about the outcome – doesn't actually change the probability.

We already saw an example like this…

Conditioning Practice

Red die 6 conditioned on sum 7 1/6

Red die 6 conditioned on sum 9 1/4

Sum 7 conditioned on red die 6 1/6

Red die 6 has probability 1/6 before or after conditioning on sum 7.

	D2=1	D2=2	D2=3	D2=4	D2=5	D2=6
D1=1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1.6)
D1=2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
D1=3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
D1=4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
D1=5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
D1=6	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

Independence

Independence

Two events A, B are independent if
$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$$

You'll sometimes see this called "statistical independence" to emphasize that we're talking about probabilities (not, say, physical interactions).

If A, B both have non-zero probability then

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B) \Leftrightarrow \mathbb{P}(A|B) = \mathbb{P}(A) \Leftrightarrow \mathbb{P}(B|A) = \mathbb{P}(B)$$

Examples

We flip a fair coin three times. Each flip is independent. (both in the statistical independence sense and in the "doesn't affect the next one" sense).

Is $E = \{HHH\}$ independent of $F =$"at most two heads"?

Are $A =$"the first flip is heads" and $B =$"the second flip is tails" independent?

Examples

Is $E = \{HHH\}$ independent of $F =$"at most two heads"?

$\mathbb{P}(E \cap F) = 0$ (can't have all three heads and at most two heads).

$\mathbb{P}(E) = 1/8$, $\mathbb{P}(F) = 7/8$, $\mathbb{P}(E \cap F) \neq \mathbb{P}(E)\mathbb{P}(F)$. These are dependent!

Are $A =$"the first flip is heads" and $B =$"the second flip is tails" independent?

$\mathbb{P}(A \cap B) = 2/8$ (uniform measure, and two of eight outcomes meet both A and B.

$\mathbb{P}(A) = 1/2$, $\mathbb{P}(B) = 1/2$; $\frac{2}{8} = \frac{1}{2} \cdot \frac{1}{2}$. These are independent!

Hey Wait

I said “the flips are independent” why aren’t E, F independent?

“the flips are independent” means events like <the first flip is blah>” is independent of events like <the second flip is blah>

But if you have an event that involves both flip one and two that might not be independent of an event involving flip one or two.

Mutual Exclusion and independence

Two of these statements are true, one is false. Explain to each other which ones are true, and find a counter-example to the false one.

1. If A, B both have nonzero probability and they are mutually exclusive, then they cannot be independent.

2. If A has zero probability, then A, B are independent (for any B).

3. If two events are independent, then at least one has nonzero probability.

Fill out the poll everywhere so Robbie knows how long to explain
Go to pollev.com/avk5

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Chain Rule

Chain Rule

We defined conditional probability as: $\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$

Which means $\mathbb{P}(A \cap B) = \mathbb{P}(A|B)\mathbb{P}(B)$

Chain Rule

$$\mathbb{P}(A_1 \cap A_2 \cap \cdots \cap A_n)$$
$$= \mathbb{P}(A_n|A_1 \cap \cdots \cap A_{n-1}) \cdot \mathbb{P}(A_{n-1}|A_1 \cap \cdots \cap A_{n-2}) \cdots \mathbb{P}(A_2|A_1) \cdot \mathbb{P}(A_1)$$

Chain Rule Example

Shuffle a standard deck of 52 cards (so every ordering is equally likely).

Let A be the event "The top card is a K♦"

Let B be the event "the second card is a J ♠

Let C be the event "the third card is a 5 ♠

What is $\mathbb{P}(A \cap B \cap C)$?

Use the chain rule!

$\mathbb{P}(A) \cdot \mathbb{P}(B|A) \cdot \mathbb{P}(C|A \cap B)$

$\frac{1}{52} \cdot \frac{1}{51} \cdot \frac{1}{50}$

Conditional Independence

Conditional Independence Example

You have two coins. Coin A is fair, coin B comes up heads with probability 0.85.

You will roll a (fair) die, if the result is odd flip coin A twice (independently); if the result is even flip coin B twice (independently)

Let C_1 be the event "the first flip is heads", C_2 be the event "the second flip is heads", O be the event "the die was odd"

Are C_1 and C_2 independent? Are they independent conditioned on O?

Conditional Independence

We say A and B are conditionally independent on C if

$$\mathbb{P}(A \cap B|C) = \mathbb{P}(A|C) \cdot \mathbb{P}(B|C)$$

i.e. if you condition on C, they are independent.

(Unconditioned) Independence

$$\mathbb{P}(C_1) = \mathbb{P}(O)\mathbb{P}(C_1|O) + \mathbb{P}(\bar{O})\mathbb{P}(C_1|\bar{O})$$

$$= \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot 0.85 = .675$$

$\mathbb{P}(C_2) = .675$ (the same formula works)

$$\mathbb{P}(C_1)\mathbb{P}(C_2) = .675^2 = .455625$$

$$\mathbb{P}(C_1 \cap C_2) = \mathbb{P}(O)\mathbb{P}(C_1 \cap C_2|O) + \mathbb{P}(\bar{O})\mathbb{P}(C_1 \cap C_2|\bar{O})$$

$$= \frac{1}{2} \cdot \frac{1}{4} + \frac{1}{2} \cdot .85^2 = .48625$$

$$.455625 \neq .48625$$

Those aren't the same! They're not independent!

Conditional Independence

$\mathbb{P}(C_1|O) = 1/2$

$\mathbb{P}(C_2|O) = 1/2$

$\mathbb{P}(C_1 \cap C_2|O) = \frac{1}{2} \cdot \frac{1}{2} = 1/4$

$\mathbb{P}(C_1|O)\mathbb{P}(C_2|O) = \mathbb{P}(C_1 \cap C_2|O)$

Yes! C_1 and C_2 are conditionally independent, conditioned on O.

Takeaway

Read a problem carefully – when we say "these steps are independent of each other" about some part of a sequential process, it's usually "conditioned on all prior steps, these steps are conditionally independent of each other."

Those conditional steps are usually dependent (without conditioning) because they might give you information about which branch you took.

More Independence

Independence of events

Recall the definition of independence of **events**:

Independence

Two events A, B are independent if

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$$

Independence for 3 or more events

For three or more events, we need two kinds of independence

Pairwise Independence

Events $A_1, A_2, \ldots, A_n$ are pairwise independent if
$\mathbb{P}(A_i \cap A_j) = \mathbb{P}(A_i) \cdot \mathbb{P}(A_j)$ for all i, j

Mutual Independence

Events $A_1, A_2, \ldots, A_n$ are mutually independent if
$\mathbb{P}(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}) = \mathbb{P}(A_{i_1}) \cdot \mathbb{P}(A_{i_2}) \cdots \mathbb{P}(A_{i_k})$
for every subset $\{i_1, i_2, \ldots, i_k\}$ of $\{1, 2, \ldots, n\}$.

Pairwise Independence vs. Mutual Independence

Roll two fair dice (one red, one blue) independently

$R =$"red die is 3"

$B =$"blue die is 5"

$S =$"sum is 7"

Pairwise Independence

R, B, S are pairwise independent

$\mathbb{P}(R \cap B)\ ?= \mathbb{P}(R)\mathbb{P}(B)$

$\frac{1}{36} = \frac{1}{6} \cdot \frac{1}{6}$ Yes! (These are also independent by the problem statement)

$\mathbb{P}(R \cap S)\ ?= \mathbb{P}(R)\mathbb{P}(S)$

$\frac{1}{36}\ ?= \frac{1}{6} \cdot \frac{1}{6}$ Yes!

$\mathbb{P}(B \cap S)\ ?= \mathbb{P}(B)\mathbb{P}(S)$

$\frac{1}{36}\ ?= \frac{1}{6} \cdot \frac{1}{6}$ Yes!

Mutual Independence

R, B, S are not mutually independent.

$\mathbb{P}(R \cap B \cap S) = 0$; if the red die is 3, and blue die is 5 then the sum is 8 (so it can't be 7)

$$\mathbb{P}(R)\mathbb{P}(B)\mathbb{P}(S) = \left(\frac{1}{6}\right)^3 = \frac{1}{216} \neq 0$$

Checking Mutual Independence

It's not enough to check just $\mathbb{P}(A \cap B \cap C)$ either.

Roll a fair 8-sided die.

Let A be $\{1,2,3,4\}$

B be $\{2,4,6,8\}$

C be $\{2,3,5,7\}$

$\mathbb{P}(A \cap B \cap C) = \mathbb{P}(\{2\}) = \frac{1}{8}$

$\mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$

Checking Mutual Independence

It's not enough to check just $\mathbb{P}(A \cap B \cap C)$ either.

Roll a fair 8-sided die.

Let A be $\{1,2,3,4\}$

B be $\{2,4,6,8\}$

C be $\{2,3,5,7\}$

$\mathbb{P}(A \cap B \cap C) = \mathbb{P}(\{2\}) = \frac{1}{8}$

$\mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$

But B and C aren't independent. Because there's a subset that's not independent, A, B, C are not mutually independent.

Checking Mutual Independence

To check mutual independence of events:

Check **every** subset.

To check pairwise independence of events:

Check **every** subset of size two.

Why Two Versions?

Pairwise independence is often all you need and is easier to design an experiment/code to achieve it.

"Pairwise independent hash functions" are a theoretical example.

Mutual Independence would let us vastly simplify the chain rule computation.

$P(A_1 \cap \cdots \cap A_n) = \mathbb{P}(A_1) \cdot \mathbb{P}(A_2|A_1) \cdot \mathbb{P}(A_3|A_2 \cap A_1) \cdots \mathbb{P}(A_n|A_1 \cap \cdots \cap A_{n-1})$

Simplifies to $\mathbb{P}(A_1) \cdot \mathbb{P}(A_2) \cdot \mathbb{P}(A_3) \cdots \mathbb{P}(A_n)$