Wrap Up Counting
Probability Definitions

CSE 312 Summer 25
Lecture 4

Outline

Today

Counting Practice

Formal Definitions of Probability

We've seen lots of ways to count

Sum rule (split into disjoint sets)

Product rule (use a sequential process)

Combinations (order doesn't matter)

Permutations (order does matter)

Principle of Inclusion-Exclusion

Complementary Counting

"Stars and Bars" $\binom{n+k-1}{k-1}$

Niche Rules (useful in very specific circumstances)

- Binomial Theorem
- Pigeonhole Principle

Extra Practice

Cards

A "standard" deck of cards has 52 cards. Each card has a suit

diamonds ♦,

hearts ♥,

clubs ♧,

spades ♠

and a value (Ace,2,3,4,5,6,7,8,9,10,Jack,Queen, King).

A "5-card-hand" is a set of 5 cards

How many five-card "flushes" are there? – a flush is a hand of cards all of the same suit.

Cards

$4 \cdot \binom{13}{5}$ $\binom{13}{5} \cdot 4$

How many five-card "flushes" are there? – a flush is a hand of cards all of the same suit.

Way 1: How can I describe a flush? Which suit it is, and which values:

$\binom{4}{1} \cdot \binom{13}{5}$

Cards

Way 2: Pretend order matters. The first card can be anything,

After that, you'll have 12 options (the remaining cards of the suit), then 11, ...

Then divide by $5!$, since order isn't supposed to matter.

$$\frac{52 \cdot 12 \cdot 11 \cdot 10 \cdot 9}{5!}$$

This is the same number as what we got on the last slide!

Fixing The Overcounting

A Solution with a Problem

You wish to count the number of 5-card hands with at least 3 aces.

There are 4 Aces (and 48 non aces)

$\binom{4}{3} \cdot \binom{49}{2}$

Choose the three aces. Then of the 49 remaining cards (the last ace is allowed as well, because we're allowed to have all 4)

What's wrong with this calculation?

What's the right answer?

A Solution with a Problem

For a hand, there should be exactly one set of choices in the sequential process that gets us there.

{A♧, A♠, A♦} {A♥, K ♠}

And

{A♧, A♠, A♥}, {A♦, K ♠}

Are two different choices of the process, but they lead to the same hand!

A Solution with a Problem

We could count exactly which hands appear more than once, and how many times each appears and compensate for it.

An easier solution is to try again…

The problem was trying to account for the "at least" – come up with disjoint sets and count separately.

$$\binom{4}{3} \cdot \binom{48}{2} + \binom{4}{4} \cdot \binom{48}{1}$$

If there are exactly 3 aces, we choose which 3 of the 4, then choose which 2 cards among the 48 non-aces. If all 4 aces appear, then one of the remaining 48 cards finishes the hand. Applying the sum rule completes the calculation.

When do we overcount?

If there are exactly 4 Aces in the hand, then we count the hand 4 different times (once for each ace as an "extra" one:

{A♣, A♠, A♦}, {A♥, ?}

{A♣, A♠, A♥}, {A♦, ?}

{A♣, A♥, A♦}, {A♠, ?}

{A♥, A♠, A♦}, {A♣, ?}

How much do we overcount?

There are 48 such hands (one for every card that could be "?" on the last slide)

So we've counted $3 \cdot 48$ processes that shouldn't count.

That would give a corrected total of $\binom{4}{3} \cdot \binom{49}{2} - 3 \cdot 48$

This is the same number as we got with our other counting.

Takeaway

It's hard to count sets where one of the conditions is "at least X"

You usually need to break those conditions up into disjoint sets and use the sum rule.

Another Problem

You have to choose 8 pieces of fruit. There are apples, oranges, and bananas.

You need to pick at most 2 apples and at least 1 banana. How many sets of fruit can you choose?

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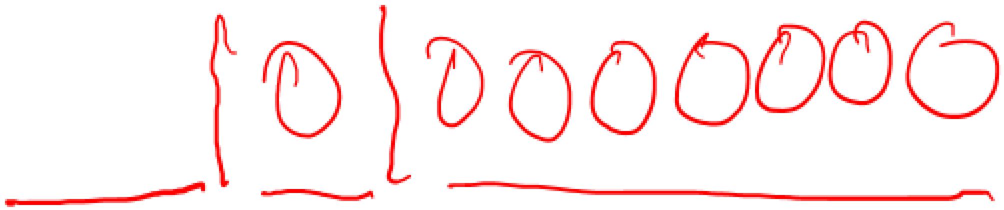
Divide into cases based on number of apples:

0 apples: 1 to 8 bananas possible (8 options)

1 apple: 1 to 7 bananas possible (7 options)

2 apples: 1 to 6 bananas possible (6 options)

21 total (by sum rule)



Another Problem

You have to choose 8 pieces of fruit. There are apples, oranges, and bananas. You need to pick at most 2 apples and at least 1 banana. How many sets of fruit can you choose?

Pick out your first banana. Problem is now to pick 7 fruits (at most 2 apples, allowed to take apples oranges and bananas)

Ignore apple restriction, and subtract off when too many apples:

Ignore restriction: $\binom{7+3-1}{3-1}$

≥ 3 apples, $\binom{4+3-1}{3-1}$ (choose 3 apples first, pick 4 remaining)

Total: $\binom{9}{2} - \binom{6}{2} = 36 - 15 = 21$

Takeaways

For stars and bars style problems with "twists", it sometimes helps to "just throw the first few in the box" to get a problem that is exactly in the stars and bars framework.

When you can do a problem two **very** different ways and get the same answer, you get much more confident in the answer.

Practice

How do we know which rule to apply?

With practice you can pick out patterns for which ones might be plausible.

But if as you're working you realize things are getting out of control, put it aside and try something different.

Which Tool Do I Use?

Pick k things from universe of n $(n \geq k)$	Repetition is NOT allowed	Repetition IS allowed
Order does NOT matter	Combinations $\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$	Stars and Bars $\binom{n+(k-1)}{k-1}$
Order does matter	Permutations $P(n,k) = \frac{n!}{(n-k)!}$	Product rule $n \cdot n \cdots n = n^k$

This is **NOT** foolproof! Sometimes you need a twist on the formula; sometimes it's a completely different tool. But a sign where to start.

Probability

Probability

Probability is a way of quantifying our uncertainty.

When more than one outcome is possible,

To have "real-world" examples, we'll need to start with some foundational processes that we're going to assert exist

- We can flip a coin, and each face is equally likely to come up
- We can roll a die, and every number is equally likely to come up
- We can shuffle a deck of cards so that every ordering is equally likely.

Sample Space

Sample Space

A sample space Ω is the set of all possible outcomes of an experiment.

"outcome" is our word for a single element of Ω.

Examples:

For a single coin flip, $\Omega = \{H, T\}$

For a series of two coin flips, $\Omega = \{HH, HT, TH, TT\}$

For rolling a (normal) die: $\Omega = \{1,2,3,4,5,6\}$

Events

Event

An event $E \subseteq \Omega$ is a subset of possible outcomes (i.e. a subset of Ω)

Examples:

Get at least one head among two coin flips ($E = \{HH, HT, TH\}$) TT

Get an even number on a die-roll ($E = \{2,4,6\}$).

Examples

I roll a blue 4-sided die and a red 4-sided die.

The table contains the sample space.

The event "the sum of the dice is even" is in gold

The event "the blue die is 1" is in green

	D2=1	D2=2	D2=3	D2=4
D1=1	(1,1)	(1,2)	(1,3)	(1,4)
D1=2	(2,1)	(2,2)	(2,3)	(2,4)
D1=3	(3,1)	(3,2)	(3,3)	(3,4)
D1=4	(4,1)	(4,2)	(4,3)	(4,4)

Probability

$\omega \in \Omega$

Probability

A probability is a number between 0 and 1 describing how likely a particular outcome is.

We'll define a function

$\mathbb{P}: \Omega \to [0,1]$

i.e. $\mathbb{P}$ takes an element of Ω as input and outputs the probability of the outcome.

We'll also use $\Pr[\omega]$, $P(\omega)$ as notation.

Example

$P(H) = P(T) = \frac{1}{2}$

Imagine we toss one coin.

Our sample space $\Omega = \{H, T\}$

What do you want $\mathbb{P}$ to be?

Example

Imagine we toss one coin.

Our sample space $\Omega = \{H, T\}$

What do you want $\mathbb{P}$ to be?

It depends on what we want to model

If the coin is fair $\mathbb{P}(H) = \mathbb{P}(T) = \frac{1}{2}$.

But we also might have a biased coin: $\mathbb{P}(H) = .85$, $\mathbb{P}(T) = 0.15$.

Probability Space

Probability Space

A (discrete) probability space is a pair $(\Omega, \mathbb{P})$ where:

Ω is the sample space

$\mathbb{P}: \Omega \to [0, 1]$ is the probability measure.

$\mathbb{P}$ satisfies:

- $\mathbb{P}(x) \geq 0$ for all x
- $\sum_{x \in \Omega} \mathbb{P}(x) = 1$
- If $E, F \subseteq \Omega$ and $E \cap F = \emptyset$ then $\mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F)$

Probability Space

Flip a fair coin and roll a fair (6-sided) die.

$\Omega = \{H, T\} \times \{1,2,3,4,5,6\}$

$\mathbb{P}(\omega) = \frac{1}{12}$ for every $\omega \in \Omega$

Is this a valid probability space?

$\mathbb{P}$ takes in elements of Ω and outputs numbers between 0 and 1

$\sum_{\omega \in \Omega} \mathbb{P}(\omega) = 1$.

Measure

$\Omega = \{H, T\} \times \{1,2,3,4,5,6\}$

$\mathbb{P}(\omega) = \frac{1}{12}$ for every $\omega \in \Omega$

So what's the probability of seeing a heads?

Seeing heads isn't an element of the sample space!

Define $\mathbb{P}(E) = \sum_{\omega \in \mathrm{E}} \mathbb{P}(\omega)$

Mutually Exclusive Events

Two events E, F are mutually exclusive if they cannot happen simultaneously.

In notation, $E \cap F = \emptyset$ (i.e. they're disjoint subsets of the sample space).

For example, if $\Omega = \{H, T\} \times \{1,2,3,4,5,6\}$

E_1 = "the coin came up heads"

E_2 = "the coin came up tails"

E_3 = "the die showed an even number"

E_1 and E_2 are mutually exclusive.
E_1 and E_3 are not mutually exclusive.

More Examples!

Suppose you roll two dice. Each die is fair and they don't affect each other. What is the probability of both dice being even?

What is your sample space?

What is your probability measure $\mathbb{P}$?

What is your event?

What is the probability?

More Examples!

Suppose you roll two dice. Each die is fair and they don't affect each other. What is the probability of both dice being even?

What is your sample space? $\{1,2,3,4,5,6\} \times \{1,2,3,4,5,6\}$

What is your probability measure $\mathbb{P}$? $\mathbb{P}(\omega) = 1/36$ for all $\omega \in \Omega$

What is your event? $\{2,4,6\} \times \{2,4,6\}$

What is the probability? $3^2/6^2$

More Examples!

Suppose you roll two dice. Each die is fair and they don't affect each other. What is the probability of both dice being even?

What if we can't tell the dice apart and always put the dice in increasing order by value.

What is your sample space?

$\{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),(2,2),(2,3),(2,4),(2,5),(2,6)$

$(3,3),(3,4),(3,5),(3,6),(4,4),(4,5),(4,6),(5,5),(5,6),(6,6)\}$

What is your probability measure $\mathbb{P}$?

$\mathbb{P}((x,y)) = 2/36$ if $x \neq y$, $\mathbb{P}(x,x) = 1/36$

What is your event? $\{(2,2),(4,4),(6,6),(2,4),(2,6),(4,6)\}$

What is the probability? $3 \cdot \frac{1}{36} + 3 \cdot \frac{2}{36} = \frac{9}{36}$

Takeaways

There is often more than one sample space possible! But one is probably easier than the others.

Finding a sample space that will make the uniform measure correct will usually make finding the probabilities easier to calculate.