Even More Counting

CSE 312 Summer 25
Lecture 3

Quiz 1

Announcements

HW 1 is out! Due Wednesday.

Office hours started yesterday! Visit now before others start on the homework.

Also, please start on the homework early. Remember you need to have a meaningful attempt to utilize the penalty free late days.

Outline

So Far

Sum and Product Rules

Combinations (order doesn’t matter) and Permutations (order does matter)

Introduce ordering and remove it to make calculations easier

Some Proofs by counting two ways

Binomial Theorem

This Time

Principle of Inclusion-Exclusion

Pigeonhole Principle

Stars and Bars

Principle of Inclusion-Exclusion

Example

How many length 5 strings over the alphabet $\{a, b, c, \ldots, z\}$ contain:

Exactly 2 'a's OR

Exactly 1 'b' OR

No 'x's

For what A, B, C do we want $|A \cup B \cup C|$?

Why not just use the sum rule?

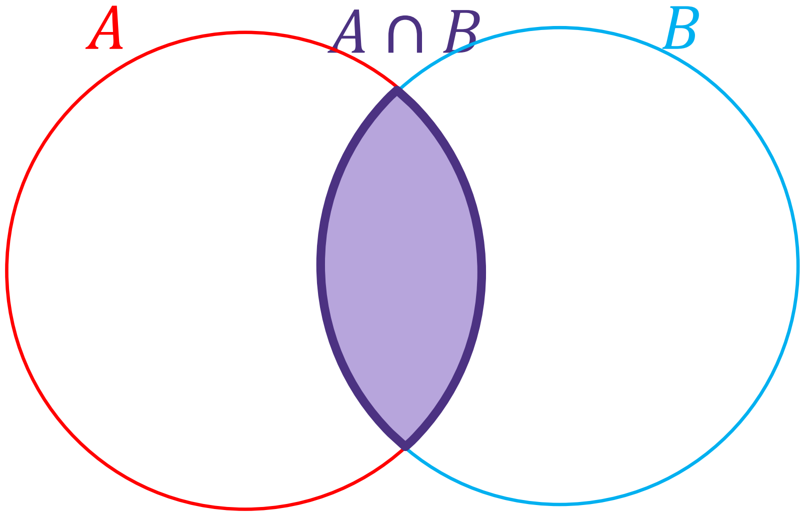
Principle of Inclusion-Exclusion

The sum rule says when A and B are disjoint (no intersection), then $|A \cup B| = |A| + |B|$.

What about when A and B aren't disjoint?

For two sets:

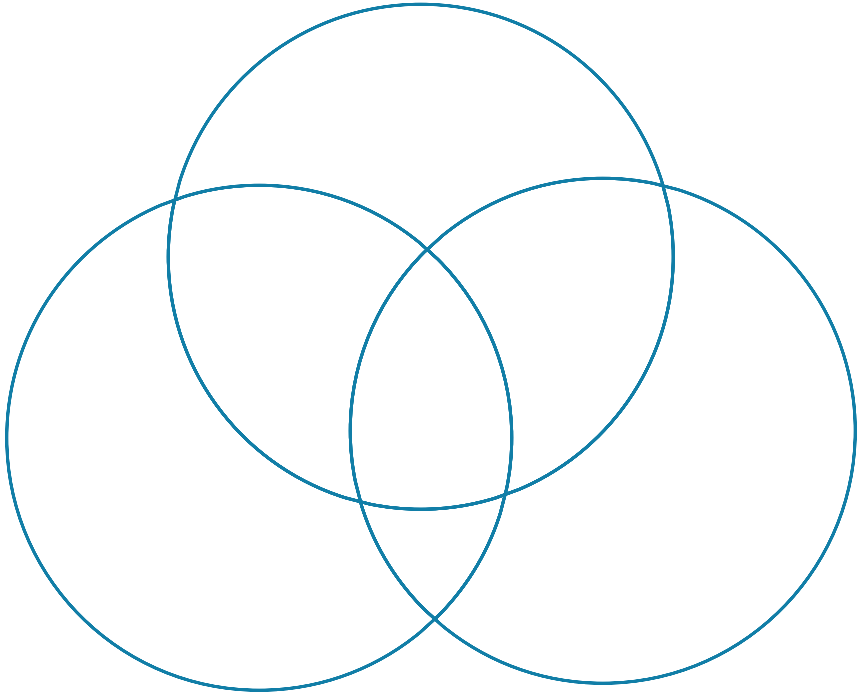
$|A \cup B| = |A| + |B| - |A \cap B|$



Principle of Inclusion-Exclusion

For three sets:

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$$



In general:

$$|A_1 \cup A_2 \cup \cdots \cup A_n| =$$

$$|A_1| + |A_2| + \cdots + |A_n|$$

$$-(\,|A_1 \cap A_2| + |A_1 \cap A_3| + \cdots + |A_1 \cap A_n| + |A_2 \cap A_3| + \cdots + |A_{n-1} \cap A_n|)$$

$$+(\,|A_1 \cap A_2 \cap A_3| + \cdots + |A_{n-2} \cap A_{n-1} \cap A_n|)$$

$$-\ \ldots$$

$$+(-1)^{n+1}|A_1 \cap A_2 \cap \cdots \cap A_n|$$

Add the individual sets, subtract all pairwise intersections, add all three-wise intersections, subtract all four-wise intersections,…, [add/subtract] the n-wise intersection.

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No ‘x’s

For what A, B, C do we want $|A \cup B \cup C|$?

Example

How many length 5 strings over the alphabet $\{a, b, c, \ldots, z\}$ contain:
Exactly 2 'a's OR
Exactly 1 'b' OR
No 'x's

$A =$ {length 5 strings that contain exactly 2 'a's}

$B =$ {length 5 strings that contain exactly 1 'b's}

$C =$ {length 5 strings that contain no 'x's }

$|A| = \binom{5}{2} \cdot 25^3$ (need to choose which "spots" are 'a' and remaining string)

$|B| = \binom{5}{1} \cdot 25^4$

$|C| = 25^5$

Example

How many length 5 strings over the alphabet $\{a, b, c, \ldots, z\}$ contain:
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B = {length 5 strings that contain exactly 1 ‘b’s}
C = {length 5 strings that contain no ‘x’s }

$|A| = \binom{5}{2} \cdot 25^3$
$|B| = \binom{5}{1} \cdot 25^3$
$|C| = 25^5$

$|A \cap B| = \binom{5}{2} \cdot \binom{3}{1} \cdot 24^2$ (choose ‘a’ spots, ‘b’ spot, remaining chars)
$|A \cap C| = \binom{5}{2} \cdot 24^3$ (choose ‘a’ spots, remaining [non-‘x’] chars)
$|B \cap C| = \binom{5}{1} \cdot 24^4$
$|A \cap B \cap C| = \binom{5}{2} \cdot \binom{3}{1} \cdot 23^2$ (choose ‘a’ spots, ‘b’ spot, remaining [non-‘x’] chars)

Example

How many length 5 strings over the alphabet $\{a, b, c, \dots, z\}$ contain:
Exactly 2 'a's OR
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$|A \cap B| = \binom{5}{2} \cdot \binom{3}{1} \cdot 24^2$

$|A \cap C| = \binom{5}{2} \cdot 24^3$

$|B \cap C| = \binom{5}{1} \cdot 24^4$

$|A \cap B \cap C| = \binom{5}{2} \cdot \binom{3}{1} \cdot 23^2$

$$|A \cup B \cup C| =$$
$$= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$
$$= \binom{5}{2} \cdot 25^3 + \binom{5}{1} \cdot 25^4 + 25^5 - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$
$$= 11{,}875{,}000 - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$
$$= 11{,}875{,}000 - \binom{5}{2} \cdot \binom{3}{1} \cdot 24^2 - \binom{5}{2} \cdot 24^3 - \binom{5}{1} \cdot 24^4 + |A \cap B \cap C|$$
$$= 11{,}875{,}000 - 1{,}814{,}400 + |A \cap B \cap C|$$
$$= 10{,}060{,}600 + |A \cap B \cap C|$$
$$= 10{,}060{,}600 + \binom{5}{2} \cdot \binom{3}{1} \cdot 23^2$$
$$= 10{,}060{,}600 + 15{,}870$$
$$= 10{,}076{,}470$$

Practical tips

Give yourself clear definitions of A, B, C.

Make a table of all the formulas you need before you start actually calculating.

Calculate “size-by-size” and incorporate into the total.

Basic check: If (in an intermediate step) you ever:

1. Get a negative value
2. Get a value greater than the prior max by adding (after all the single sets)
3. Get a value less than the prior min by subtracting (after all the pairwise intersections)

Then something has gone wrong.

Pigeonhole Principle

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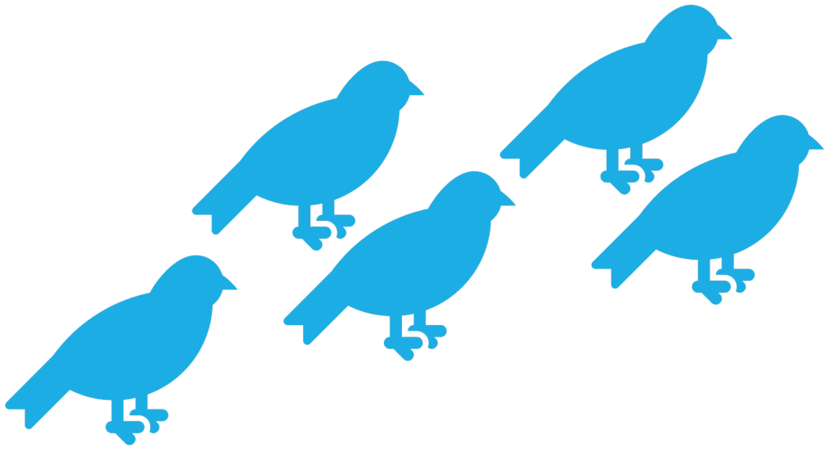
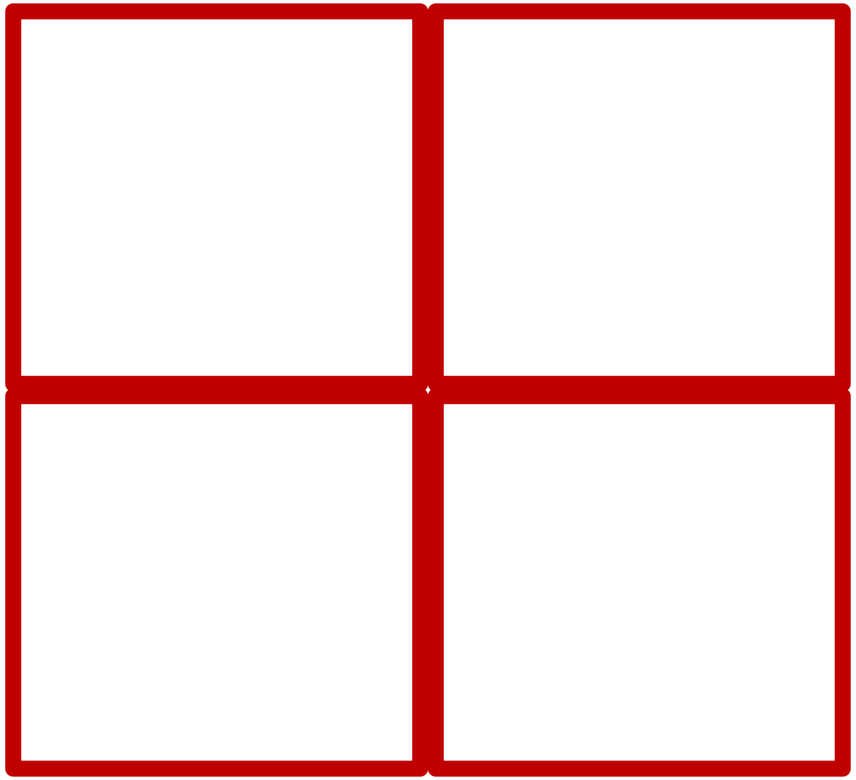
If you have 5 pigeons, and place them into 4 holes, then…

At least two pigeons are in the same hole.



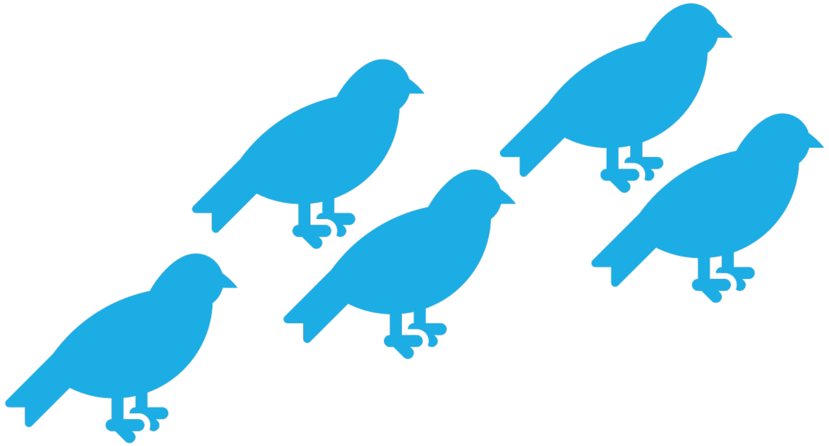


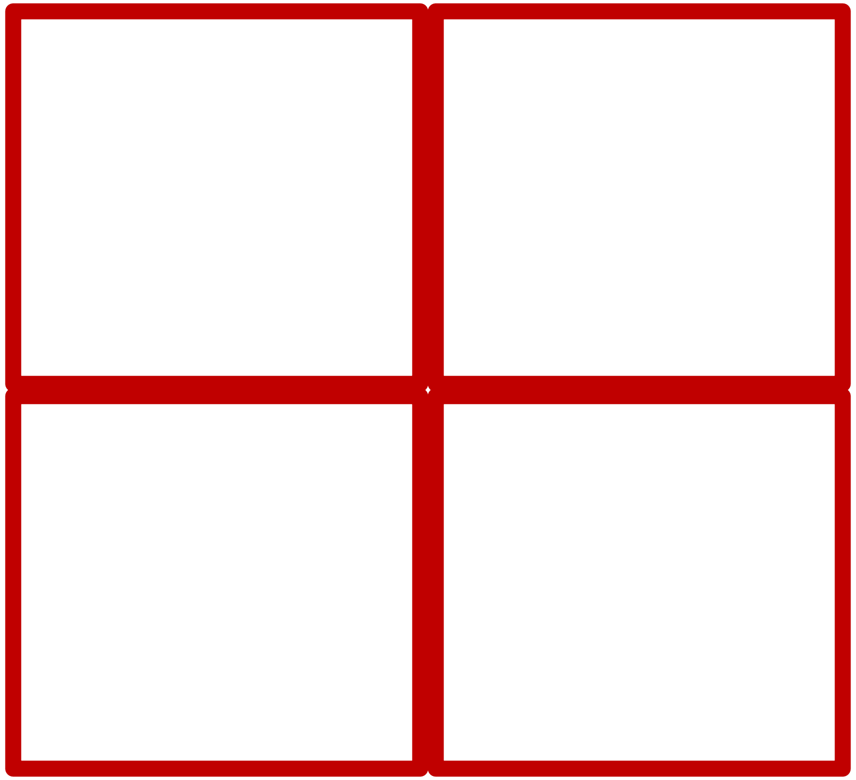
Pigeonhole Principle

If you have 5 pigeons, and place them into 4 holes, then…

At least two pigeons are in the same hole.

It might be more than two.





Strong Pigeonhole Principle

If you have n pigeons and k pigeonholes, then there is at least one pigeonhole that has at least $\left\lceil \frac{n}{k} \right\rceil$ pigeons.

$\lceil a \rceil$ is the "ceiling" of a (it means always round up, $\lceil 1.1 \rceil = 2$, $\lceil 1 \rceil = 1$).

An example

If you have to grow 10 crops, and you have 3 seasons to grow them in, then…

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Pigeons: The crops 

Pigeonholes: The seasons 

Mapping: Which crop you grow in which season.

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If you have to grow 10 crops, and you have 3 seasons to grow them in, then…

Pigeons: The crops

Pigeonholes: The seasons

Mapping: Which crop you grow in which season.

Applying the (generalized) pigeonhole principle, there is at least one season where you will grow at least $\left\lceil \frac{10}{3} \right\rceil = 4$ crops.

Practical Tips

When the pigeonhole principle is the right tool, it's usually the first thing you'd think of or the absolute last thing you'd think of.

For **really** tricky ones, we'll warn you in advance that it's the right method (you'll see one in the section handout).

When applying the principle, say:

- What are the pigeons
- What are the pigeonholes
- How do you map from pigeons to pigeonholes

Look for – a set you're trying to divide into groups, where collisions would help you somehow.

Stars and Bars

One More Counting Rule

It's the start of spring and you want to plant a 3x4 plot on your farm.

You're going to go to Pierre's General Store to buy twelve seeds.

There are parsnip, cauliflower, potato, kale, and garlic seeds (i.e. five types of seeds)

How many different 12 seed orders can you make?

Consider two orders the same if they contain the same number of every type of seed (order doesn't matter).

One More Counting Rule

You’re going to go to Pierre's General Store to buy twelve seeds.

There are parsnip, cauliflower, potato, kale, and garlic seeds (i.e. five types of seeds)

Put seeds in order by type, then put dividers between the types.

Explanation 1

Think of it as a string.

There are $12 + (5 - 1)$ characters.

But 12 are the "seed" character (identical) and 4 are the "divider" character (identical).

So?

$$\frac{16!}{12!4!}$$

i.e., $\binom{16}{4}$

Placing Dividers

Place a divider – how many possible locations are there?

13 – before seed 1, before 2, …, before seed 12, after seed 12.

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Place the second divider, how many possible locations are there?

14 – one of the previous spots was split ("before" and "after" the last divider)

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Place the second divider, how many possible locations are there?

14 – one of the previous spots was split (“before” and “after” the first divider)

In general, placing divider i has $12 + i$ possible locations.

Wrapping Up

We had 12 seeds, how many dividers do we need?

4 (to divide into 5 groups)

Count so far: $13 \cdot 14 \cdot 15 \cdot 16$

Are we done?

Wrapping Up

Count so far: $13 \cdot 14 \cdot 15 \cdot 16$

This count treats all dividers as different – they're not! Divide by $4!$.

For n seeds of k types

$$\frac{(n+1)(n+2)\cdots(n+k-1)}{(k-1)!}$$

That's a combination! $\binom{n+k-1}{k-1}$

Wrapping Up

$\binom{n+k-1}{k-1}$

We wrote down a “string” consisting of n  and $k-1$ |

$n+k-1$ characters, n “seeds” are identical, $k-1$ “dividers” are identical, so divide by the rearrangements (like we did for RHUBARB).

In General

To pick n objects from k groups (where order doesn't matter and every element of each group is indistinguishable), use the formula:

$$\binom{n+(k-1)}{k-1}$$

The counting technique we did is often called "stars and bars"

using a "star" instead of a seed shape, and calling the dividers "bars"