More Counting

CSE 312 Summer 25
Lecture 2

Announcements

Homework 1 will come out tonight, due Wed July 2nd.

Mostly written problems.

Every written problem requires a **justification.**

Not a proof (unless we say to prove something). But enough explanation that someone who followed lecture but hasn't seen the problem would fully understand where your answer came from (and believe it's correct)

There's also one programming question – submission on gradescope, but it will be easiest to do the questions on Ed (Ed has a python interpreter built in, you could set up your own python environment, but Ed will be sufficient for this quarter).

Announcements

You’ll have 2 late days to use per HW assignment (except the last one) if you submit a meaningful attempt by the original deadline. Failure to submit a meaningful attempt will result in 25% penalty per day.

This policy is intended for “normal” things during the quarter.

If you have an unusual or extended or extreme issue, please let us know.

The sooner you let us know, the more options we have for accommodations.

Announcements

Please read the collaboration policy, in particular:
on homeworks you are **encouraged** to collaborate, but the writeup must be your own.

- If you're working with others, don't take notes from your discussions
- And take a 30 minute break between discussions and your writeup

The goal of these rules is to make sure you've learned how to do the problems.

ChatGPT (or any other AI system) is **not** a valid collaborator.

- Our goal is that you learn how to do the problems, not that ChatGPT learns how to do the problems.

Announcements

Office hours will start tomorrow.

Check the course website of times and locations.

Where Are We?

Last time:

Sum and Product Rules

Sequential Processes

Different ways of thinking lead to different formulas

Today:

A few more rules: Combinations and Permutations.

Proof by double counting

Binomial Theorem

Factorial

That formula shows up a lot.

The number of ways to "permute" (i.e. "reorder", i.e. "list without repeats") n elements is "n factorial"

n factorial

$$n! = n \cdot (n-1) \cdot (n-2) \cdots 1$$

We only define $n!$ for natural numbers n.

As a convention, we define: $0! = 1$.

One More Counting Technique

Complementary Counting

Count the complement of the set you're interested in.

How many length 5 strings over $\{a, b, c, \ldots, z\}$ are there with **at least** 1 'a'

Let A be the set of strings we're interested in, $\mathcal{U}$ be all length 5 strings

$|A| = |\mathcal{U} \setminus \bar{A}| = |\mathcal{U}| - |\bar{A}| = 26^5 - 25^5$

Distinct Letters

How many length 5 strings are there over the alphabet $\{a, b, \ldots, z\}$ where each string does not repeat a letter.

E.g. "morel"  is an allowed string, but "berry" is not, nor is "crocus" 

$26 \cdot 25 \cdot 24 \cdot 23 \cdot 22$

In General

k-permutation

The number of k-element sequences of distinct symbols from a universe of n symbols is:

$$P(n,k) = n \cdot (n-1) \cdots (n-k+1) = \frac{n!}{(n-k)!}$$

Said out loud as “P n k” or “n permute k” or “n pick k”

Alternative notation: ${}_nP_k$

Edge cases: $P(n,n) = n!$, $P(n,0) = 1$,
$P(n,k)$ for $k < 0$ or $k > n$ is undefined.

Change it slightly

How many **subsets** of size 5 are there of $\{a, b, \dots, z\}$

Remember subsets we don't count repeats – so we still have that rule.

But for subsets order doesn't matter.

$\{m, o, r, e, l\}$ is the same set as $\{m, o, l, e, r\}$ (even though "morel" and "moler" are different strings).

Number of Subsets

k-combination

The number of k-element subsets from a set of n symbols is:

$$C(n,k) = \frac{P(n,k)}{k!} = \frac{n!}{k!\,(n-k)!}$$

Said out loud “n choose k” (or sometimes: “n combination k”)

Lots of notation:

${}_nC_k$ or $\binom{n}{k}$ or $C(n,k)$ all mean “number of size-k subsets of a size-n set.”

Edge cases: $\binom{n}{0} = 1, \binom{n}{n} = 1$; $\binom{n}{k}$ for $k < 0$ or $k > n$ is undefined.

Clever approach – count two ways

Let's artificially introduce a requirement that we are supposed to have an ordered list.

Then the total is going to be $P(26,5)$.

How else could we get an ordered list? With this sequential process:

Step 1: Choose a subset.

Step 2: Put the subset in order.

These better give us the same number, so:

$$\frac{26!}{(26-5)!} = ? \cdot 5!$$

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So the number of size-5 subsets of a size-26 set is:

$$\frac{26!}{(26-5)!\,5!}$$

Second Takeaway

The second way of counting hints at a generally useful trick:

Pretend that order does matter, then divide by the number of orderings of the parts where order doesn't matter.

For example, here's another way to get the formula for combinations:

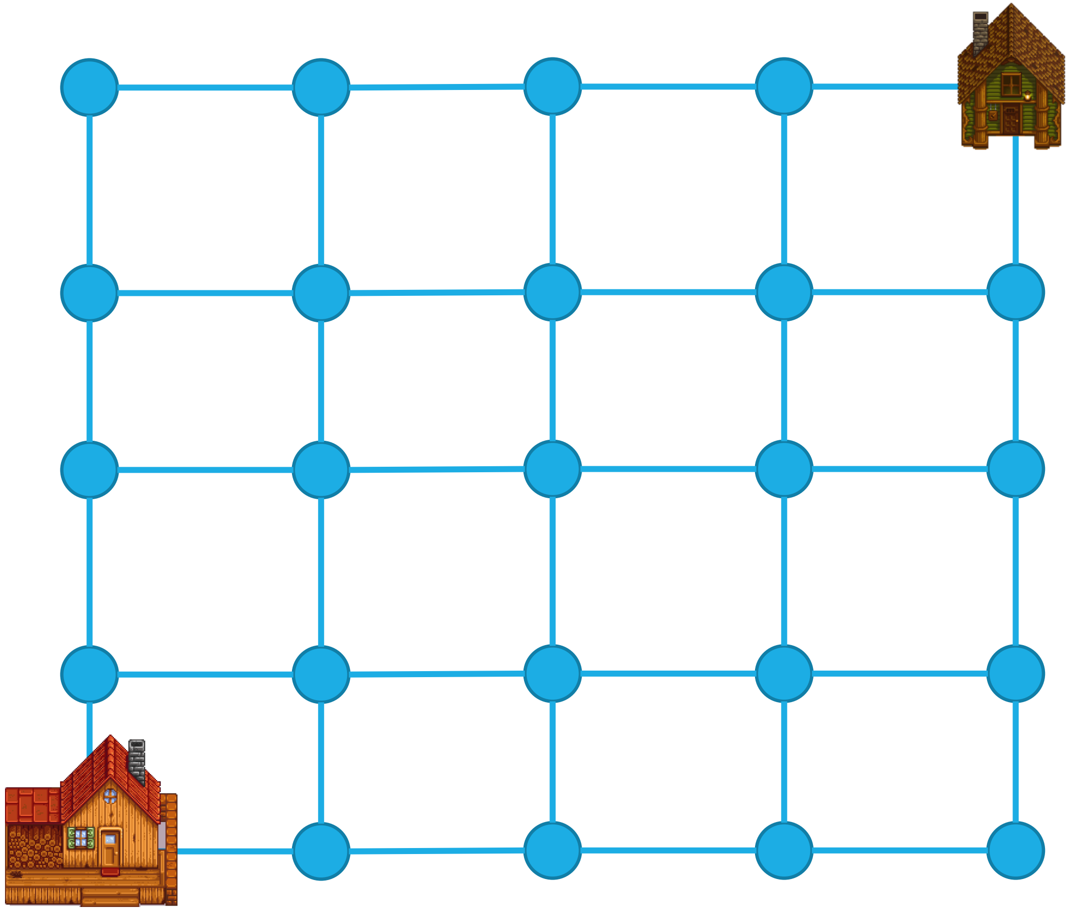
You have n elements. Put them in order, take the first k as your set.

$n!$ Orderings overall. We've overcounted because:

- Among the first k, order doesn't matter between them. Divide by $k!$.
- Among the last $n - k$, order doesn't matter between them. Divide by $(n - k)!$.

$$\frac{n!}{k!\,(n-k)!}$$

Path Counting



We're at our farm in the lower-left corner and want to get to the Adventure's Guild in the upper-right corner.

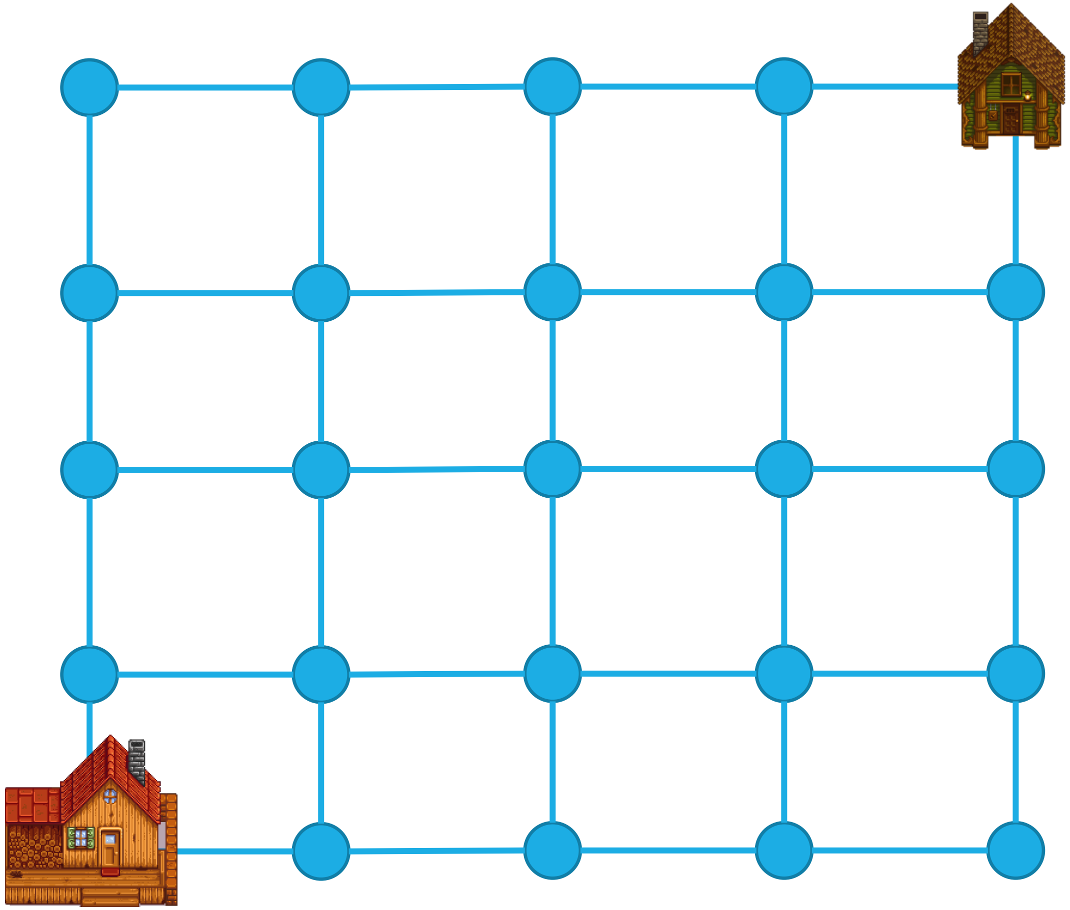
We're only going to go right and up.

How many different paths are there?

A. 2^8
B. $P(8,4)$
C. $\binom{8}{4}$
D. Something else

https://pollev.com/avk5

Path Counting



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Idea 1:
We're going to take 8 steps.
Choose which SET of 4 of the steps will be up (the others will be down).

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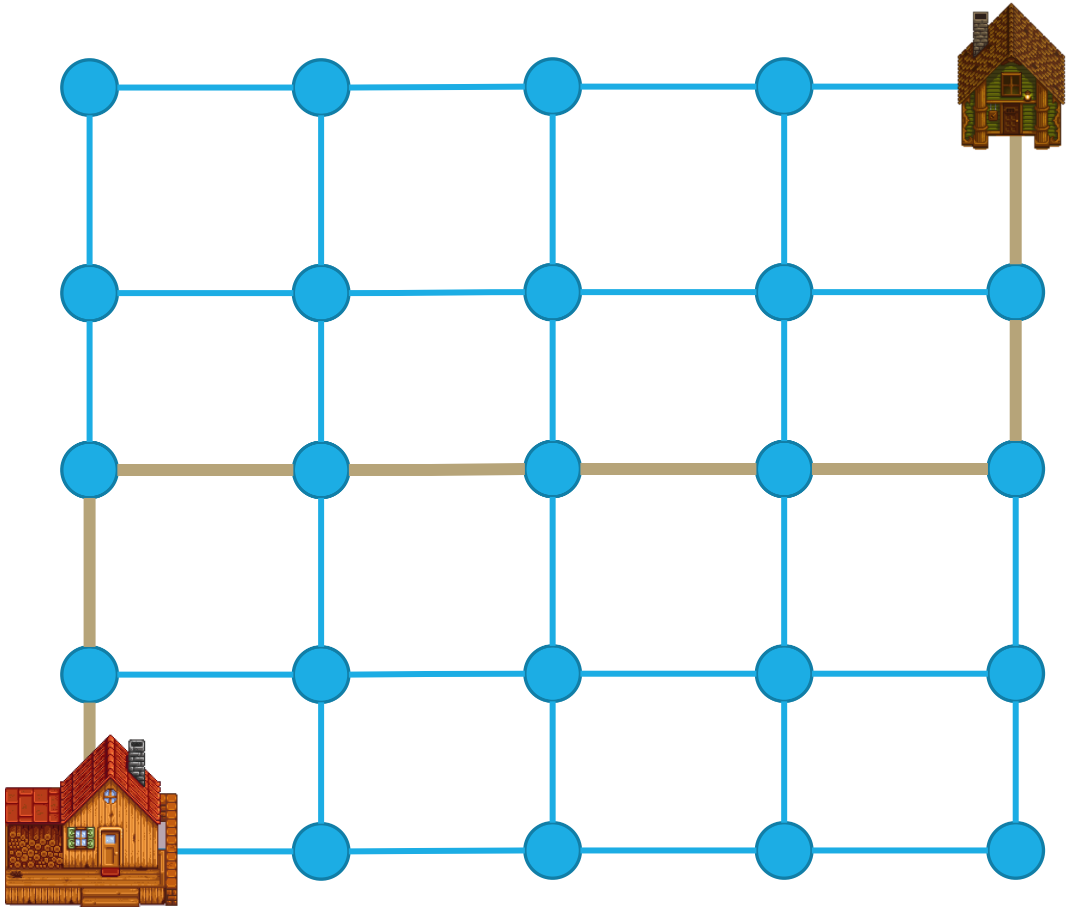

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E.g. $\{1,2,7,8\}$ is
How many size-4 subsets of $\{1,2,3,4,5,6,7,8\}$ are there?

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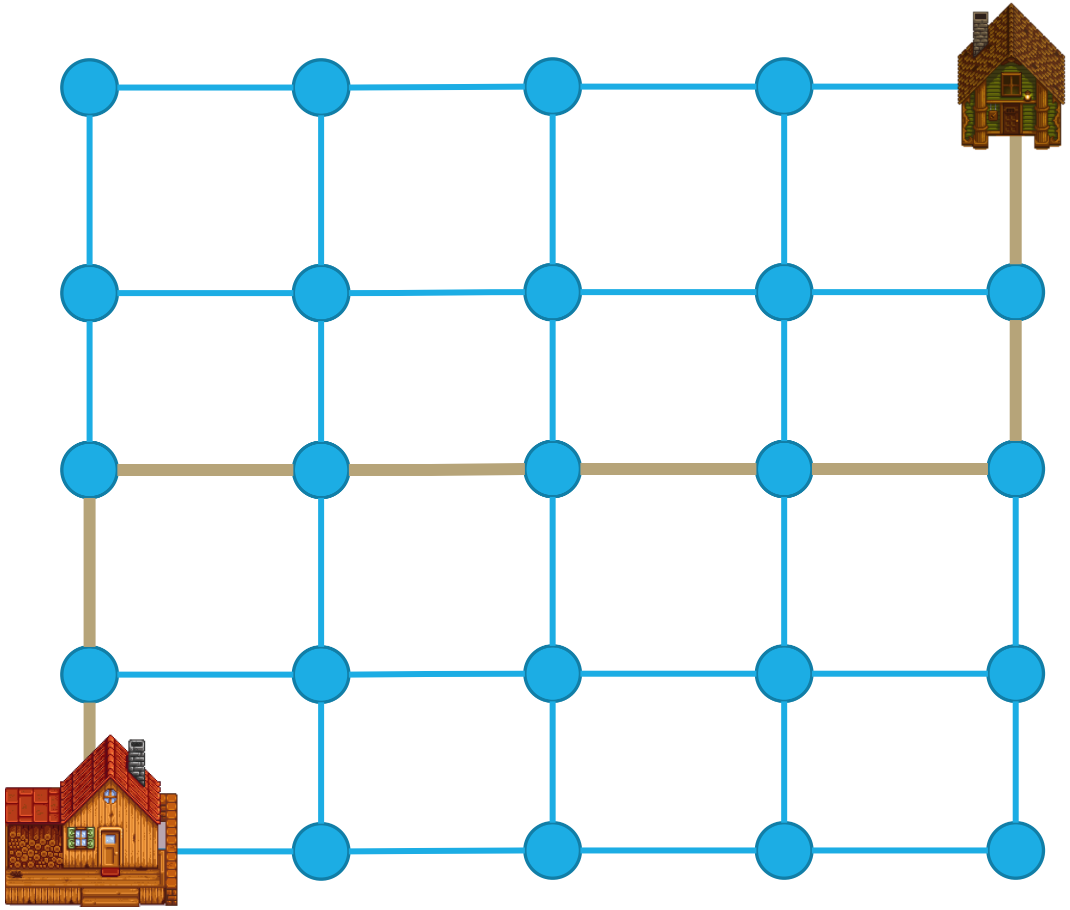

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$\binom{8}{4}$ is the answer.

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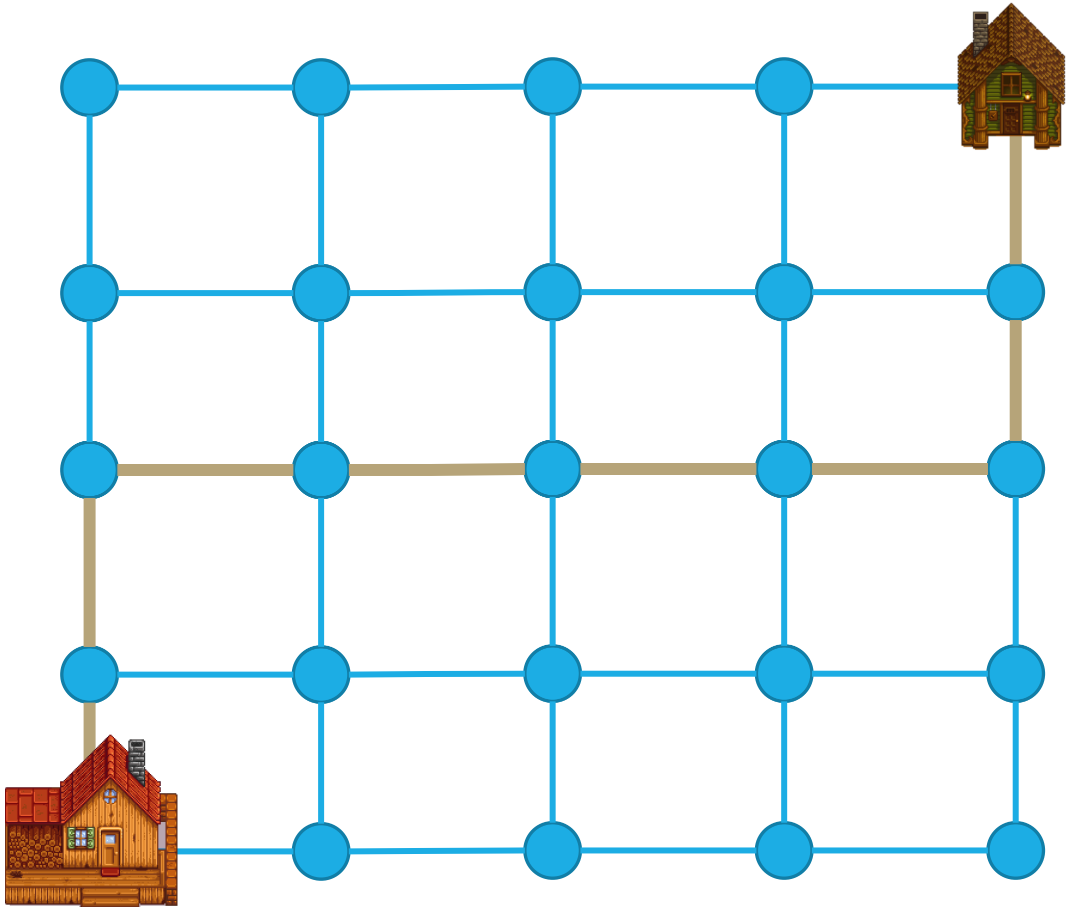

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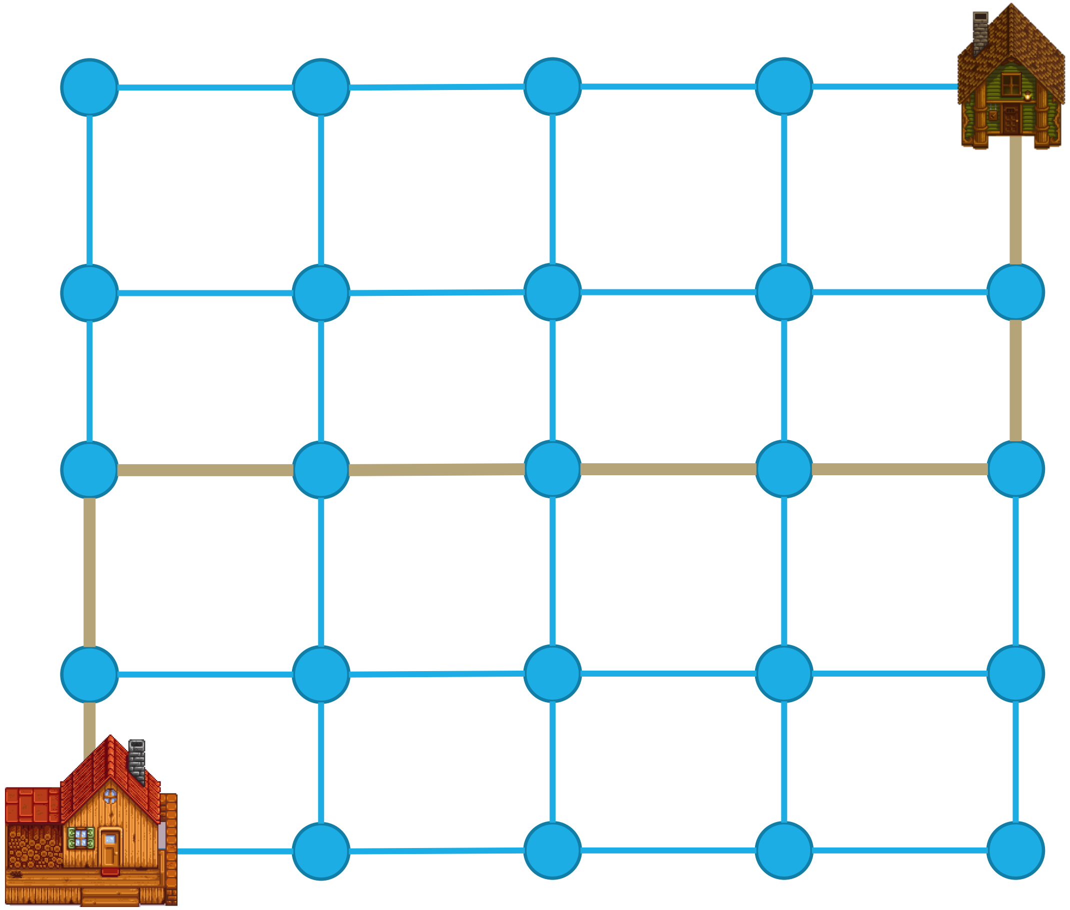
How many different paths are there?

Idea 2: Introduce artificial ordering

Order $\uparrow_A \uparrow_B \uparrow_C \uparrow_D \rightarrow_A \rightarrow_B \rightarrow_C \rightarrow_D$ 8!

Path Counting



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Idea 2: Introduce artificial ordering

Order $\uparrow_A \uparrow_B \uparrow_C \uparrow_D \rightarrow_A \rightarrow_B \rightarrow_C \rightarrow_D$ $\quad 8!$

Remove the overcounting

Those $4 \uparrow$ are really the same, divide by $4!$

The $4 \rightarrow$ are really the same, divide by $4!$

Total: $\frac{8!}{4! \cdot 4!}$

$\binom{8}{4}$ is the answer.

Overcounting

How many anagrams are there of RHUBARB 

(an anagram is a rearrangement of letters).

It’s not 7! That counts RHUBARB and RHUBARB as different things!

I swapped the Rs (or maybe the Bs)

Overcounting

How many anagrams are there of RHUBARB 

Pretend the order of the Rs (and Bs) relative to each other matter (that RHUBARB and RHUBARB are different)

How many arrangements of RHUBARB? $7!$

How have we overcounted? Rs relative to each other and Bs relative to each other $2! \cdot 2!$

Final answer $\frac{7!}{2! \cdot 2!}$

Overcounting

How many anagrams are there of RHUBARB 

Final answer $\frac{7!}{2!\cdot 2!}$

One more piece of notation – "multinomial coefficient"

$\binom{7}{2,2}$ is alternate notation for $\frac{7!}{2!2!}$.

In general: $\binom{n}{k_1,k_2,\ldots,k_\ell} = \frac{n!}{k_1!\cdot k_2!\cdots k_\ell!}$

Popular notation among mathematicians.

Combination Facts

Some Facts about combinations

Symmetry of combinations: $\binom{n}{k} = \binom{n}{n-k}$

Pascal's Rule: $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

Two Proofs of Symmetry

Proof 1: By algebra

$\binom{n}{k} = \frac{n!}{k!(n-k)!}$ Definition of Combination

$= \frac{n!}{(n-k)!k!}$ Algebra (commutativity of multiplication)

$= \binom{n}{n-k}$ Definition of Combination

Two Proofs of Symmetry

Wasn’t that a great proof.

Airtight. No disputing it.

Got to say “commutativity of multiplication.”

But…do you know *why*? Can you *feel* why it’s true?

Two Proofs of Symmetry

Suppose you have n people, and need to choose k people to be on your team. We will count the number of possible teams two different ways.

Way 1: We choose the k people to be on the team. Since order doesn't matter (you're on the team or not), there are $\binom{n}{k}$ possible teams.

Way 2: We choose the $n-k$ people to NOT be on the team. Everyone else is on it. Since order again doesn't matter, there are $\binom{n}{n-k}$ possible ways to choose the team.

Since we're counting the same thing, the numbers must be equal.
So $\binom{n}{k} = \binom{n}{n-k}$.

Pascal's Rule: $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

$\binom{n-1}{k-1} + \binom{n-1}{k} = \dfrac{(n-1)!}{(k-1)!(n-1-[k-1])!} + \dfrac{(n-1)!}{k!(n-1-k)!}$	definition of combination
$= \dfrac{(n-1)!}{(k-1)!(n-k)!} + \dfrac{(n-1)!}{k!(n-k-1)!}$	subtraction
$= \dfrac{[(n-1)!k!(n-k-1)!] + [(n-1)!(k-1)!(n-k)!]}{k!(k-1)!(n-k)!(n-k-1)!}$	Find a common denominator
$= \dfrac{(n-1)!(k-1)!(n-k-1)!\,[k+(n-k)]}{k!(k-1)!(n-k)!(n-k-1)!}$	factor out common terms
$= \dfrac{(n-1)!\,[k+(n-k)]}{k!(n-k)!}$	Cancel $(k-1)!(n-k-1)!$
$= \dfrac{(n-1)! \cdot n}{k!(n-k)!} = \dfrac{n!}{k!(n-k)!}$	Algebra
$= \binom{n}{k}$	Definition of combination

Pascal's Rule: $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

You and $n-1$ other people are trying out for a k person team. How many possible teams are there?

Way 1: There are n people total, of which we're choosing k (and since it's a team order doesn't matter) $\binom{n}{k}$.

Way 2: There are two types of teams. Those for which you make the team, and those for which you don't.

If you do make the team, then $k-1$ of the other $n-1$ also make it.

If you don't make the team, k of the other $n-1$ also make it.

Overall, by sum rule, $\binom{n-1}{k-1} + \binom{n-1}{k}$.

Since we're computing the same number two different ways, they must be equal. So: $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

Takeaways

Formulas for factorial, permutations, combinations.

A useful trick for counting is to pretend order matters, then account for the overcounting at the end (by dividing out repetitions)

When trying to prove facts about counting, try to have each side of the equation count the same thing.

- Much more fun and much more informative than just churning through algebra.

Binomial Theorem

Binomial Theorem

In high school you probably memorized

$(x + y)^2 = x^2 + 2xy + y^2$

And $(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

The Binomial Theorem tells us what happens for every n:

The Binomial Theorem

$$(x + y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$$

Some intuition

The Binomial Theorem

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$$

Intuition: Every monomial on the right-hand-side has either x or y from each of the terms on the left.

How many copies of $x^i y^{n-i}$ do you get? Well how many ways are there to choose i x's and $n-i$ y's? $\binom{n}{i}$.

Formal proof? Induction!

So What?

Well…if you saw it before, now you have a better understanding now of why it's true.

There are also a few cute applications of the binomial theorem to proving other theorems (usually by plugging in numbers for x and y) – you'll do one on HW1.

For example, set $x = 1$ and $y = 1$ then

$2^n = (1+1)^n = \sum_{i=0}^{n} \binom{n}{i} 1^i 1^{n-i} = \sum_{i=0}^{n} \binom{n}{i}$.

i.e. if you sum up binomial coefficients, you get 2^n. Exercise: reprove this equation (directly) with a combinatorial proof (where have we seen 2^n recently?)

Extra Practice

Books, revisited

Remember the books problem from lecture 1? Books 1,2,3,4,5 need to be assigned to Leo, Jas, and Vincent (each book to exactly one person).

Now that we know combinations, try a sequential process approach. It won't be as nice as the change of perspective, but we can make it work.

Break into cases based on how many books Leo gets, use the sum rule to combine.

Books, revisited

Step 1: give Leo gets 0 books (1 way to do this)

Step 2: give Jas a subset of the remaining books 2^5 ways.

Step 3: give Vincent the remaining books (no choice – 1 way)

+

Step 1: give Leo 1 book ($\binom{5}{1}$ ways to do this)

Step 2: give Jas a subset of the 4 remaining books 2^4 ways.

Step 3: give Vincent the remaining books (no choice – 1 way)

+ …

Books, revisited

Add all the options together

$$1 \cdot 2^5 \cdot 1 + \binom{5}{1} \cdot 2^4 \cdot 1 + \binom{5}{2} \cdot 2^3 \cdot 1 + \binom{5}{3} \cdot 2^2 \cdot 1 + \binom{5}{4} \cdot 2^1 \cdot 1 + \binom{5}{5} \cdot 2^0 \cdot 1$$

If you plug and chug, you’ll get the number we got last time. It took quite a bit of work, but we got there!