Task 2 – Multiple Choice/Short answer [27 pts]

Questions with ◯ have exactly one correct answer. **Fully fill in** the one with the best answer.

a) A lock has a sequence of 4 dials, each labeled with digits from 0 to 9. How many unique sequences are possible if each dial sequence must a **strictly increasing** sequence (i.e., each digit must be greater than the digit preceding it)? For example, 1234, 0678, 3579 are some possible outcomes. (5 points)

◯ $\binom{10}{4}$
◯ $P(10, 4)$
◯ 10^4
◯ 4^{10}

$\binom{10}{4}$. Since we want the sequence to be strictly increasing, we need to pick 4 distinct elements. For every distinct subset of 4 numbers, there's only 1 way to order them so that they are in a strictly increasing order.

b) You have an overflowing drawer of mismatched socks. Each sock is one of 4 colors. What is the smallest number of socks you need to draw to guarantee that you will have at least one pair of "matching" socks with the same color? (4 points)

◯ 3
◯ 4
◯ 5
◯ 9

By the pigeonhole principle, if you pick 5 socks, at least 2 will definitely fall into the same hole and have the same color.

c) If $\mathbb{P}(A) = 0.4$ and $\mathbb{P}(B) = 0.3$, what is $\mathbb{P}(A \cup B)$ if A and B are independent events? (5 points)

By inclusion exclusion, $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$. Because A and B are independent, $\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B) = 0.4 \cdot 0.3$. So, $\mathbb{P}(A \cup B) = 0.4 + 0.3 - 0.4 \cdot 0.3 = 0.58$

d) Run the following experiment: flip a fair coin. If it is heads, roll a fair 6-sided die twice (independently); if it is tails, roll a fair 20-sided die twice (independently). (5 points)
Let A be the event you roll a 10 on the first roll.
Let B be the event you roll a 10 on the second roll.
Let T be the event the coin comes up tails.

Mark **all** true statements below (one or more statements might be true):

☐ A and B are independent.
☐ A and B are conditionally independent, conditioned on T.
☐ A and B are conditionally independent, conditioned on $\overline{T}$.

The second and third statements are true. A and B are NOT independent because, for example, knowing A means the 20-sided die was rolled, and the probability of B would increase. If we know the 6-sided dice was rolled, both events have probability 0, so they are independent (2nd statement). If we know the 20-sided dice was rolled, both dice are rolled independently (3rd statement).

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e) You are attending a concert where tickets cost X dollars each (X is a random variable), and you plan to buy 10 tickets. Additionally, there is a fixed cost of 9 dollars for other expenses such as food. What is the **expected value** of the total cost? (4 points)

- ◯ 90
- ◯ $10\mathbb{E}[X]$
- ◯ $10\mathbb{E}[X] + 9$
- ◯ $10^2 \cdot \mathbb{E}[X]$

By linearity of expectation, $\mathbb{E}[10X + 9] = 10\mathbb{E}[X] + 9$

.

f) Is the following a valid PMF? (4 points)

$$p_X(k) = \begin{cases} \frac{3}{4} & k = 1 \\ \frac{1}{4} & k \in \{2, 3\} \\ 0 & \text{otherwise} \end{cases}$$

- ◯ Yes
- ◯ No
- ◯ Not enough information

No, because the probabilities do not sum up to 1. $p_X(1)+p_X(2)+p_X(3) = \frac{3}{4}+\frac{1}{4}+\frac{1}{4} = \frac{5}{4} \neq 1$.

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Task 3 – 312's Got Talent! [30 pts]

10 people are organizing a talent show. Unfortunately, they've all been so busy and can't think of any individual talents to showcase. So, they decide to collaborate on 3 acts: a hula-hooping choir, animal impersonating, and spoon balancing. For each act, they will randomly select 3 people to work on that act. Any subset of 3 people is equally likely to be chosen for each act, and people can be assigned to multiple acts.

a) How many possible assignments are there? In other words, how many possible ways are there to select a subset of 3 distinct people for each act (people can be assigned to any number of acts)? Order within the people in each act does not matter. (6 points)

Each act has $\binom{10}{3}$ options for the 3 people chosen for that act. Since we choose 3 people for each act, there are $\binom{10}{3}^3$ options in total.

b) Two of the people in this group are Mario and Luigi! A person is overworked if they are assigned to ALL of the three acts. What is the **probability** that either Mario **or** Luigi is overworked? You may write your answer in terms of q, the correct answer to part (a). (10 points)
You MUST show your work for this part b. You don't need to write in the full level of detail as the homeworks, but show your work/rules applied at a high level.

Our sample space is the set of all possible assignments of people to the tasks, so the size of the sample space is Ω. The event is the event that Mario or Luigi is overworked. Let M and L be the set of outcomes where Mario and Luigi are overworked respectively. The size of the event is $|M \cup S| = |M| + |S| - |M \cap S|$ by inclusion exclusion. $|M| = |S| = \binom{9}{2}^3$ because since Mario/Luigi is working in all 3 acts, we need to pick 2 more from the remaining 3 for all 3 acts. $|M \cap S| = 8^3$ because once we pick both Mario and Luigi for each of the 3 acts, we pick one of the remaining 8 people for each of the 3 acts. So, the probability is $\frac{\binom{9}{2}^3 + \binom{9}{2}^3 - 8^3}{q}$.

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The problem from the last page continues below, here is the problem statement again:
10 people are organizing a talent show. Unfortunately, they've all been so busy and can't think of any individual talents to showcase. So, they decide to collaborate on three acts: a hula-hooping choir, animal impersonating, and spoon balancing. For each act, they will randomly select 3 people to work on that act. Any subset of 3 people is equally likely to be chosen for each act, and people can be assigned to multiple acts.

c) Someone found 12 **identical** rubber chickens that will be perfect to elevate all three of these acts! How many ways are there to distribute these identical rubber chickens among these 3 acts? (6 points)

Since we're distributions 12 identical chickens among the 3 acts, we use stars and bars with 12 stars and 3 bins: $\binom{12+3-1}{3-1} = \binom{12+3-1}{12}$

d) Every distribution of these 12 identical rubber chickens is equally likely. What is the **probability** the hula-hooping choir act gets at least 4 rubber chickens? (rubber chickens are great at singing) (8 points)
You may use m in your expression to represent the correct answer to the previous part (c).

We have a uniform probability space, where every distribution in the sample space is equally likely. From the previous part, $|\Omega| = m$. For the size of the event, we first give the choir act 4 rubber chickens, and then distribute the remaining 8 chickens among the 3 acts. There are $\binom{8+3-1}{3-1}$ options for this. So, the probability is $\frac{\binom{10}{2}}{m}$.

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Task 4 – Crumbling Probabilities [19 pts]

Claris loves to bake (some) cookies! But she is very good at burning cookies in the oven. The probability of her burning a batch of cookies depends on what kind of cookies she is making:

- If she is making sugar cookies the probability of her burning the batch is 0.8.
- If she is making chocolate chip cookies, probability of burning the batch is 0.5.
- If she is making brownie cookies, she keeps a very close eye, and the probability of her **successfully** baking (i.e., **not** burning) the batch is 0.6.

Claris only ever bakes brownie, sugar, or chocolate chip cookies, and she never mixes cookies of different types in the same batch.

Today, Claris is feeling very indecisive so she rolls a fair 3-sided dice to determine which one of the three cookie types to make a batch of, so she is equally likely to picky any of the three cookie types to make. Let B, S, and C be the events that she makes brownie, sugar, and chocolate chip cookies respectively. Let N be the event that she burns the batch of cookies.

a) What probability is being referred to in the last bullet point in terms of the events defined above? (3 points)
Fill in the appropriate symbols below:

$$\mathbb{P}\left(\overline{N}|B\right)$$

b) What is the probability that Claris burns the batch of cookies? (8 points)

$$\mathbb{P}\left(N\right) = \mathbb{P}(N|B)\mathbb{P}(B) + \mathbb{P}(N|S)\mathbb{P}(S) + \mathbb{P}(N|C)\mathbb{P}(C) = 0.8 \cdot \frac{1}{3} + 0.5 \cdot \frac{1}{3} + 0.4 \cdot \frac{1}{3}$$

c) You walk by and smell a strong burnt smell. What is the probability that Claris is making sugar cookies if you know that the batch is burned? You may write your answer in terms of m, the *correct* answer to part (b). (8 points)

$$\mathbb{P}\left(S|N\right) = \frac{\mathbb{P}(N|S)\mathbb{P}(()S)}{\mathbb{P}(()N)} = \frac{0.8 \cdot \frac{1}{3}}{m}$$

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Task 5 – Truffle Shuffle [23 pts]

In a cooking competition, there are 10 unique ingredients, one of which is a rare type of "truffle oil". 2 chefs each randomly and independently pick 2 distinct ingredients to create their dishes. Multiple chefs can pick the same ingredients. Let X be the random variable representing the number of chefs who pick the truffle oil as one of their ingredients.

a) What is the support/range of X? (3 points)

The support is $\Omega_X = \{0, 1, 2\}$.

b) What is the PMF of X? (4 points)

$$P(X = k) = \begin{cases} \left(\frac{\binom{9}{2}}{\binom{10}{2}}\right)^2 & k = 0 \\ 2 \cdot \frac{\binom{9}{1}}{\binom{10}{2}} \cdot \frac{\binom{9}{2}}{\binom{10}{2}} & k = 1 \\ \left(\frac{\binom{9}{1}\cdot\binom{1}{1}}{\binom{10}{2}}\right)^2 & k = 2 \\ 0 & \text{otherwise} \end{cases}$$

c) What is $\mathbb{E}[X]$? Compute this using the **definition of expectation**. (4 points)

$$\mathbb{E}[X] = 2 \cdot \frac{\binom{9}{1}}{\binom{10}{2}} \cdot \frac{\binom{9}{2}}{\binom{10}{2}} + 2 \cdot \left(\frac{\binom{9}{1}\cdot\binom{1}{1}}{\binom{10}{2}}\right)^2$$

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d) Now, this cooking competition is being expanded and **there are now 20 chefs**. There are still 10 distinct ingredients, and the rules are the same: Each chef randomly and independently picks 2 unique ingredients to create their dishes, and multiple chefs can pick the same ingredients.

Let Y be the number of chefs who pick the truffle as one of there ingredients. Use **linearity of expectation** to find $\mathbb{E}[Y]$. (8 points)

You MUST show your work for this part. Show how you write Y as a sum of random variables, how you apply linearity of expectation, and how you compute the expectation of each individual random variable.

Since there are 20 chefs, $X = \sum_{i=1}^{20} X_i$. By linearity of expectation, $\mathbb{E}[X] = \sum_{i=1}^{20} \mathbb{E}[X_i]$. $\mathbb{E}[X_i] = \mathbb{P}(X_i = 1) = \frac{9}{\binom{10}{2}}$. So, $\mathbb{E}[X] = \sum_{i=1}^{20} \mathbb{E}[X_i] = \sum_{i=1}^{20} \frac{9}{\binom{10}{2}} = 20 \cdot \frac{9}{\binom{10}{2}}$

e) What is Var(Y)? (4 points)

We have $Y = \sum_{i=1}^{20} Y_i$, where Y_i is the indicator that chef i picks the truffle oil.

Since the Y_i are independent (each chef picks ingredients independently),

$$\text{Var}(Y) = \sum_{i=1}^{20} \text{Var}(Y_i).$$

Now, for each i, since Y_i is an indicator random variable:

$$\text{Var}(Y_i) = \mathbb{E}[Y_i^2] - (\mathbb{E}[Y_i])^2 = \mathbb{E}[Y_i] - (\mathbb{E}[Y_i])^2,$$

because $Y_i^2 = Y_i$.
We already know $\mathbb{E}[Y_i] = \frac{9}{\binom{10}{2}} = \frac{9}{45} = \frac{1}{5}$.
Thus,

$$\text{Var}(Y_i) = \frac{1}{5} - \left(\frac{1}{5}\right)^2 = \frac{1}{5} - \frac{1}{25} = \frac{4}{25}.$$

So,

$$\text{Var}(Y) = 20 \times \frac{4}{25} = \frac{80}{25} = \frac{16}{5}.$$

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Task 6 – Grading Morale **[1 pts]**

Pitch us your idea for a new food combination or restaurant using CSE 312 concepts! (As long as this page is not empty, you will get the point)