1. Multiple Choice and True/False

For the questions below,

- Questions with ◯ Circles have exactly one best answer. **Fully fill in** the circle for the one best answer.
- Questions with ☐ squares are "mark all that apply" questions. **Fully fill in** the square for all correct options.

(a) Your friend attempts to count the number of "two pair" hands. Two pair hands contain:

- Two cards of one value (e.g., two aces or two 8's)
- Two cards of a **different** value
- A fifth card of another different value.

For a standard 52 card deck (13 values, 4 suits), your friend says the number of two pair hands is

$$13\binom{4}{2} \cdot 12\binom{4}{2} \cdot 11\binom{4}{1}.$$

Which best describes their response?

◯ It overcounts—you need to divide by $5!$ for all possible reorderings.
◯ It overcounts—you need to divide by $2!$ for reordering the "first pair" compared to the "second pair"
◯ It undercounts—you need to multiply by $5!$ for all possible reorderings.
◯ It undercounts—you need to multiply by $2!$ for reordering the "first pair" compared to the "second pair"

(b) There are m houses in a suburban neighborhood. Suppose we need to pave a (direct) path between every possible pair of houses. How many paths need to be paved? (Once paved, a path can be used in both directions).

◯ m^2 paths
◯ $\frac{m(m-1)}{2}$ paths
◯ $\frac{m}{2}$ paths
◯ $m(m-1)$ paths
◯ None of the above.

(c) Suppose X is a random variable with support $\Omega_X = \{-1, 3\}$. Mark **all** of the options which **must** be true.

☐ The expectation of X is given by: $\mathbb{E}[X] = 2$.
☐ The support of X^2 is given by: $\Omega_{X^2} = \{1, 9\}$.
☐ The variance of X is non-negative, i.e., $\text{Var}(X) \geq 0$.

(d) Let A, B, and C be events. We say that A, B, C are "3-way independent" if

$$P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C)$$

Which of the following statements must be true (mark **ALL** that apply):

☐ If A, B, and C are 3-way independent, then they are pairwise independent.
☐ If A, B, and C are pairwise independent, then they are 3-way independent.
☐ If A, B, and C are pairwise and 3-way independent, then they are mutually independent.

- Questions with ◯ Circles have exactly one best answer. **Fully fill in** the circle for the one best answer.
- Questions with ☐ squares are "mark all that apply" questions. **Fully fill in** the square for all correct options.

(e) Suppose you assign 14 books to 4 bookshelves. Which of the following is true?

◯ There is **at least** 1 bookshelf with at most 3 books
◯ There are **at least** 4 bookshelves with **at least** 3 books
◯ There is at most 1 bookshelf with **at least** 4 books
◯ There is **at least** 1 bookshelf with **at least** 4 books
◯ There are **at least** 4 bookshelves with **at least** 1 book

(f) Let A, B be events with non-zero probability, which are independent. Which of the following is true?

◯ A and B are **always** mutually exclusive
◯ A and B are **sometimes** mutually exclusive
◯ A and B are **never** mutually exclusive

(g) Consider the following attempt at writing a CDF. Is it valid?

$$F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ \lfloor x \rfloor / 20 & \text{if } 0 \leq x \leq 10 \\ \frac{1}{2} + \lfloor (x-10) \rfloor / 10 & \text{if } 10 < x \leq 15 \end{cases}$$

◯ The CDF is valid as written.
◯ The CDF requires a "0 otherwise" case.
◯ The CDF requires a "1 otherwise" case.
◯ The CDF should not have the floor functions.

(h) Run the following experiment: flip a fair coin. If it is heads, roll a fair 6-sided die twice (independently); if it is tails, roll a fair 20-sided die twice (independently).
Let A be the event you roll a 10 on the first roll.
Let B be the event you roll a 10 on the second roll.
Let T be the event the coin comes up tails.

Mark **all** true statements below:

☐ A and B are independent.
☐ A and B are conditionally independent, conditioned on T.
☐ A and B are conditionally independent, conditioned on $\overline{T}$.

2. Party Time! [16 points]

Suppose you are throwing a party and you want to buy 30 balloons. The shop has five colors: red, blue, green, purple, orange. All balloons of a given color are identical (e.g., two purple balloons are indistinguishable from each other). You are allowed to take any number of any color as long as you hit the total of 30. Let g and b be the number of green and blue balloons you take respectively.

(a) How many balloon allocations are possible (with no additional restrictions)? [4 points]

(b) How many balloon allocations are possible such that **strictly less than** 3 of the balloons are green or blue (i.e. $g + b < 3$)? [5 points]
Hint: You might want to use casework where $g + b = 0, 1$ or 2.

(c) Let a and b be the **correct** answers for parts (a) and (b) respectively. How many balloon allocations are possible such that **at least** 3 of the ballons are green or blue (i.e. $g + b \geq 3$)? Your answer **must** be in terms of a and/or b. [2 points]

(d) 40 guests come to your party and enter a lucky draw. Winners are chosen such that a gold, silver, and bronze balloon are awarded as 1st, 2nd, and 3rd prizes respectively. 7 more guests are then awarded identical consolation white balloons (and the other 30 get nothing).
How many such prize assignments are possible? [5 points]

3. Squidward's Skittles [21 points]

Squidward has a bag of 50 **distinguishable** Skittles consisting of 10 of each of the 5 original flavors: `orange`, `lemon`, `apple`[1], `grape`, and `strawberry`.

Squidward wants to spice up his Skittle-eating experience. He draws 4 random Skittles from the bag to eat as a mix of flavors. However, Squidward detests tasting certain pairs of flavors together. In particular, Squidward will be *dissatisfied* if `{apple, grape}` or `{orange, apple}` flavor combinations are subsets of the flavors in the mix of 4 Skittles.

For example, if Squidward draws `{orange, lemon, grape, grape}` then he will be satisfied, but if he draws `{orange, orange, apple, apple}` then he will be dissatisfied.

For this problem, define events:

- A to be the event that Squidward's mix does not contain any `apple` Skittles
- G to be the event that Squidward's mix does not contain any `grape` Skittles

(a) Using the events defined above, what is the event that the mix contains at least one `apple` Skittle **and** at least one `grape` Skittle? Write your answer in the line below. [3 points]

____

(b) What is the **probability** that the mix contains at least one `apple` **and** at least one `grape` Skittle? [6 points]

Hint: Use the principle of inclusion-exclusion and the event you found in part (a). If your event in part (a) has an intersection, you may want to rewrite it as an equivalent expression using a union.

[1] green apple, but apple for short

This problem continues with the same setup from the last page. Here's the setup again:

Squidward has a bag of 50 **distinguishable** Skittles consisting of 10 of each of the 5 original flavors: `orange`, `lemon`, `apple`, `grape`, and `strawberry`.

Squidward wants to spice up his Skittle-eating experience. He draws 4 random Skittles from the bag to eat as a mix of flavors. However, Squidward detests tasting certain pairs of flavors. In particular, Squidward will be *dissatisfied* if `{apple, grape}` or `{orange, apple}` flavor combinations are subsets of the flavors in the mix of 4 Skittles.

For example, if Squidward draws `{orange, lemon, grape, grape}` then he will be satisfied, but if he draws `{orange, orange, apple, apple}` then he will be dissatisfied.

For this problem, define events:

- A to be the event that Squidward's mix does not contain any `apple` Skittles
- G to be the event that Squidward's mix does not contain any `grape` Skittles

Now back to problems

(c) What is the probability that Squidward gets at least one `apple`, at least one `grape`, and at least one `orange` Skittle? [6 points]

Hint: Consider two cases where either 1) all 4 Skittles are unique flavors or 2) there is one repeated flavor amongst the 4 Skittles in the mix.

(d) Let p_1 be the correct probability obtained in part (b) and p_2 be the correct probability obtained in part (c). In terms of p_1 and p_2, what is the probability that Squidward will be *satisfied* (i.e. **not** *dissatisfied*) with his Skittle-eating experience? [6 points]

4. Shrek Goes to Section [18 points]

Donkey is unfortunately falling behind in their assignments at this point of the quarter, and against his best judgment, he is strongly considering skipping section to catch up on studying. Donkey decides whether to go to section or not by asking Shrek, a friend in the same section.

- Shrek is a very reliable friend; he goes to section with probability s.
- If Shrek goes, Donkey is motivated and decides to go to section $8/9$ of the time.
- If Shrek doesn't go, Donkey is discouraged and skips section $4/7$ of the time.

Let D be the event that Donkey goes to section, and S be the event that Shrek goes to section.

(a) Which conditional probability or probability is referred to in the **last** bullet point?
Write the appropriate symbols below: [3 points]

$$\mathbb{P}\left(\rule{3cm}{0.4pt}\right)$$

(b) What is the probability that Donkey goes to section?
Express your answer in terms of s. [5 points]

(c) If Donkey went to section, what is the probability that Shrek went to section?
Let d be the answer to part b. Express your answer in terms of d and s. [5 points]

(d) What is the probability that both Shrek and Donkey go to section?
You may express your answer in terms of previously defined variables. [5 points]

5. Random Variables [20 points]

A group of n friends is planning a book club. They have created a list of n books which they are considering reading. Each of the friends like exactly $n/2$ of the books (though which of the books could be different depending on which of the friends you ask). You may assume n is even so that $n/2$ is an integer. To form the club, each person will be given a set of 3 of the n books to read (each person gets three distinct books, but the same book can be assigned to any number of people). The set given to each person is uniformly random and independent of sets given to all others.

(a) Call a friend **very pleased** if they like **all** of the 3 books they read. What is the expected number of very pleased friends? [5 points]

(b) Once the books are assigned and read it's time to discuss! For each of the n books, every person who was assigned that book will meet to discuss. What is the expected number of books with **no** readers to discuss it? [5 points]

(c) Suppose that Edward is one of the friends. Define Y to be the number of books that Edward is assigned that he likes. Write the PMF for Y. Be sure to include all cases. [3 points]

This problem continues with the same setup from the last page. Here's the setup again:

A group of n friends is planning a book club. They have created a list of n books which they are considering reading. Each of the friends like exactly $n/2$ of the books (though which of the books could be different depending on which of the friends you ask). You may assume n is even so that $n/2$ is an integer. To form the club, each person will be given a set of 3 of the n books to read (each person gets three distinct books, but the same book can be assigned to any number of people). The set given to each person is uniformly random and independent of sets given to all others.

Now back to problems

(d) One of the books is *Probability and Statistics with Applications to Computing* (by Alex Tsun). Let X be the number of friends who are given that book.

(i) What is the support of X? [3 points]

(ii) what is Var (X)? [4 points]

6. Grading Morale [1 point]

Robbie is playing on the softball team this quarter! Draw a picture, write a poem, or make a meme of a 312 topic that is interesting to you that somehow incorporates softball.