1. Multiple Choice and True/False

For the questions below,

- Questions with ◯ Circles have exactly one best answer. **Fully fill in** the circle for the one best answer.
- Questions with ☐ squares are "mark all that apply" questions. **Fully fill in** the square for all correct options.

(a) Your friend attempts to count the number of "two pair" hands. Two pair hands contain:

- Two cards of one value (e.g., two aces or two 8's)
- Two cards of a **different** value
- A fifth card of another different value.

For a standard 52 card deck (13 values, 4 suits), your friend says the number of two pair hands is

$$13\binom{4}{2} \cdot 12\binom{4}{2} \cdot 11\binom{4}{1}.$$

Which best describes their response?

◯ It overcounts—you need to divide by $5!$ for all possible reorderings.
◯ It overcounts—you need to divide by $2!$ for reordering the "first pair" compared to the "second pair"
◯ It undercounts—you need to multiply by $5!$ for all possible reorderings.
◯ It undercounts—you need to multiply by $2!$ for reordering the "first pair" compared to the "second pair"

Solution:

It overcounts, you need to divide by $2!$. This sequential process treats (for example) $5\{H, D\}, 4\{H, D\}, 3H$ as different from $4\{H, D\}, 5\{H, D\| 3H$, but these produce the same hand, i.e. $\{5H, 5D, 4H, 4D, 3H\}$

We don't divide by $5!$, because a given hand does not correspond to $5!$ choices in the sequential process (only two reorderings).

(b) There are m houses in a suburban neighborhood. Suppose we need to pave a (direct) path between every possible pair of houses. How many paths need to be paved? (Once paved, a path can be used in both directions).

◯ m^2 paths
◯ $\frac{m(m-1)}{2}$ paths
◯ $\frac{m}{2}$ paths
◯ $m(m-1)$ paths
◯ None of the above.

Solution:

$\frac{m(m-1)}{2}$ paths. Note that $\binom{m}{2} = \frac{m(m-1)}{2}$.

We can also apply a sequential process of choosing one of m houses and then one of $m-1$ other houses for the end of the path, then dividing by 2 to account for over counting each path (since the path from A to B is the same as the path from B to A).

(c) Suppose X is a random variable with support $\Omega_X = \{-1, 3\}$. Mark **all** of the options which **must** be true.

☐ The expectation of X is given by: $\mathbb{E}[X] = 2$.
☐ The support of X^2 is given by: $\Omega_{X^2} = \{1, 9\}$.
☐ The variance of X is non-negative, i.e., $\text{Var}(X) \geq 0$.

Solution:

- False. This is only true if $\mathbb{P}(X=-1)=\mathbb{P}(X=3)=0.5$.
- True.
- True. If the variance is zero, then the support should only have 1 possible value.

(d) Let A, B, and C be events. We say that A, B, C are "3-way independent" if

$$P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C)$$

Which of the following statements must be true (mark **ALL** that apply):

☐ If A, B, and C are 3-way independent, then they are pairwise independent.
☐ If A, B, and C are pairwise independent, then they are 3-way independent.
☐ If A, B, and C are pairwise and 3-way independent, then they are mutually independent.

Solution:

(iii) only. (i) and (ii) are incorrect because there is no implication in either direction. (iii) is correct by definition of mutual independence.

(e) Suppose you assign 14 books to 4 bookshelves. Which of the following is true?

○ There is **at least** 1 bookshelf with <u>at most</u> 3 books
○ There are **at least** 4 bookshelves with **at least** 3 books
○ There is <u>at most</u> 1 bookshelf with **at least** 4 books
○ There is **at least** 1 bookshelf with **at least** 4 books
○ There are **at least** 4 bookshelves with **at least** 1 book

Solution:

There is at least 1 bookshelf with at least 4 books follows from the pigeonhole principle.

There is at least 1 bookshelf with at most 3 books is also true—to make 4 shelves with at least 3 books you need at least 16 books to assign.

(f) Let A, B be events with non-zero probability, which are independent. Which of the following is true?

○ A and B are **always** mutually exclusive
○ A and B are **sometimes** mutually exclusive
○ A and B are **never** mutually exclusive **Solution:**

never mutually exclusive.
$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$ by independence. Since A, B have nonzero probability, $\mathbb{P}(A \cap B)$ is the product of nonzero numbers, so is nonzero. By the chain rule $\mathbb{P}(A \cap B) = \mathbb{P}(B)\mathbb{P}(A|B)$, again we can deduce that $\mathbb{P}(A|B)$ is nonzero.

but $\mathbb{P}(A|B) = 0$ when they are mutually exclusive.

(g) Consider the following attempt at writing a CDF. Is it valid?

$$F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ \lfloor x \rfloor / 20 & \text{if } 0 \le x \le 10 \\ \frac{1}{2} + \lfloor (x-10) \rfloor / 10 & \text{if } 10 < x \le 15 \end{cases}$$

- ◯ The CDF is valid as written.
- ◯ The CDF requires a “0 otherwise” case.
- ◯ The CDF requires a “1 otherwise” case.
- ◯ The CDF should not have the floor functions.

Solution:

A “1 otherwise” case is required, as currently there is no value given for inputs $x > 15$. Since a CDF is cumulative, that must be 1 to be consistent with the value of 1 at $x = 15$.

(h) Run the following experiment: flip a fair coin. If it is heads, roll a fair 6-sided die twice (independently); if it is tails, roll a fair 20-sided die twice (independently).
Let A be the event you roll a 10 on the first roll.
Let B be the event you roll a 10 on the second roll.
Let T be the event the coin comes up tails.

Mark **all** true statements below:

- ☐ A and B are independent.
- ☐ A and B are conditionally independent, conditioned on T.
- ☐ A and B are conditionally independent, conditioned on $\overline{T}$. **Solution:**

A and B are conditionally indepdendent, conditioned on T and also conditioned on $\overline{T}$, since T or $\overline{T}$ tell you which die you are rolling and the rolls are then independent.

They are not independent, as the first roll will give you information about which die you picked up.

2. Party Time! [16 points]

Suppose you are throwing a party and you want to buy 30 balloons. The shop has five colors: red, blue, green, purple, orange. All balloons of a given color are identical (e.g., two purple balloons are indistinguishable from each other). You are allowed to take any number of any color as long as you hit the total of 30. Let g and b be the number of green and blue balloons you take respectively.

(a) How many balloon allocations are possible (with no additional restrictions)? [4 points] **Solution:**

Since the balloons are indistinguishable and the colors are distinguishable, we apply stars and bars. There are 30 stars (balloons) and $5-1$ bars (colors), so we have $\boxed{\binom{30+5-1}{5-1} = \binom{34}{4}}$ allocations.

(b) How many balloon allocations are possible such that **strictly less than** 3 of the balloons are green or blue (i.e. $g+b<3$)? [5 points]
Hint: You might want to use casework where $g+b=0,1$ or 2. **Solution:**

We can split this into cases and sum them up since they are disjoint. Using the same process as part (b), we have:

- $g+b=0$: 30 stars remaining, $3-1$ bars (only red, purple, orange allowed): $\binom{30+3-1}{3-1} = \binom{32}{2}$
- $g+b=1$: 2 ways to get $g+b=1$ (0,1; 1,0), 29 stars remaining so $2\binom{29+3-1}{3-1} = 2\binom{31}{2}$
- $g+b=2$: 3 ways to get $g+b=2$, 28 stars remaining so $3\binom{28+3-1}{3-1} = 3\binom{30}{2}$

Summing this up, we have $\boxed{\binom{32}{2} + 2\binom{31}{2} + 3\binom{30}{2}}$ allocations.

(c) Let a and b be the **correct** answers for parts (a) and (b) respectively. How many balloon allocations are possible such that **at least** 3 of the ballons are green or blue (i.e. $g+b \geq 3$)? Your answer **must** be in terms of a and/or b. [2 points] **Solution:**

We apply complementary counting to obtain $\boxed{a-b}$

(d) 40 guests come to your party and enter a lucky draw. Winners are chosen such that a gold, silver, and bronze balloon are awarded as 1st, 2nd, and 3rd prizes respectively. 7 more guests are then awarded identical consolation white balloons (and the other 30 get nothing).
How many such prize assignments are possible? [5 points] **Solution:**

This is a k-permutation problem with a small twist. First choose the 10 people from 40, then do $P(10,3)$ to permute the top 3 prizes. There is one way to assign identical balloons to the the remaining 7 people.
$\boxed{\binom{40}{10}\binom{10}{3}3! = \binom{40}{10}\frac{10!}{7!} = \frac{40!}{30! \cdot 7!}}$

3. Squidward's Skittles [21 points]

Squidward has a bag of 50 **distinguishable** Skittles consisting of 10 of each of the 5 original flavors: `orange`, `lemon`, `apple`[1], `grape`, and `strawberry`.

Squidward wants to spice up his Skittle-eating experience. He draws 4 random Skittles from the bag to eat as a mix of flavors. However, Squidward detests tasting certain pairs of flavors together. In particular, Squidward will be *dissatisfied* if `{apple, grape}` or `{orange, apple}` flavor combinations are subsets of the flavors in the mix of 4 Skittles.

For example, if Squidward draws `{orange, lemon, grape, grape}` then he will be satisfied, but if he draws `{orange, orange, apple, apple}` then he will be dissatisfied.

For this problem, define events:

- A to be the event that Squidward's mix does not contain any `apple` Skittles
- G to be the event that Squidward's mix does not contain any `grape` Skittles

(a) Using the events defined above, what is the event that the mix contains at least one `apple` Skittle **and** at least one `grape` Skittle? Write your answer in the line below. [3 points]

____

Solution:

$(A \cup G)^C$

Ideally, the students don't define the event as $A^C \cap G^C$ in order to better scaffold for part (b).

(b) What is the **probability** that the mix contains at least one `apple` **and** at least one `grape` Skittle? [6 points]

Hint: Use the principle of inclusion-exclusion and the event you found in part (a). If your event in part (a) has an intersection, you may want to rewrite it as an equivalent expression using a union. **Solution:**

Note that the sample space Ω represents all choices of 4 Skittles. Then $|\Omega| = \binom{50}{4}$.

We employ inclusion-exclusion. To guarantee no apple, we only select from the other flavors. Then $|A| = \binom{40}{4}$. Similarly, $|G| = \binom{40}{4}$. To get neither of the flavors, we only select from the three remaining flavors. $|A \cap G| = \binom{30}{4}$.

Then since we work under equally likely outcomes, the probability is

$$\Pr[(A \cup G)^C] = 1 - \Pr[A \cup G] = 1 - (\Pr[A] + \Pr[G] - \Pr[A \cap G]) = 1 - \left(\frac{\binom{40}{4}}{\binom{50}{4}} + \frac{\binom{40}{4}}{\binom{50}{4}} - \frac{\binom{30}{4}}{\binom{50}{4}} \right) \quad \left(= \frac{2997}{9212} \right)$$

(c) What is the probability that Squidward gets at least one `apple`, at least one `grape`, and at least one `orange` Skittle? [6 points]

Hint: Consider two cases where either 1) all 4 Skittles are unique flavors or 2) there is one repeated flavor amongst the 4 Skittles in the mix. **Solution:**

To satisfy the constraints, we must have an apple, grape, and orange Skittle. The last Skittle can either be one of the two remaining flavors or a repeat of apple, grape, or orange. Then we have two cases.

[1] green apple, but apple for short

All 4 Skittles unique: There are 2 choices for the remaining flavor and 10 choices of the Skittle from each of the 4 flavors for a total of $2 \cdot 10^4$ choices.

Only 3 Skittle flavors: There are 3 choices for the repeated flavor. For the repeated flavor, there are $\binom{10}{2}$ choices of the Skittle. For the other two flavors, there are 10 choices for the Skittle. There are thus $3 \cdot \binom{10}{2} \cdot 10^2$ total choices.

Then the probability of at least one apple, grape, and orange flavor is $\frac{2 \cdot 10^4 + 3 \cdot \binom{10}{2} \cdot 10^2}{\binom{50}{4}} \quad (= \frac{335}{2303})$.

(d) Let p_1 be the correct probability obtained in part (b) and p_2 be the correct probability obtained in part (c). In terms of p_1 and p_2, what is the probability that Squidward will be *satisfied* (i.e. **not** *dissatisfied*) with his Skittle-eating experience? [6 points]

Solution:

Note: There is a more direct approach to this problem where you take five cases based on the number of apple flavors in the mix. However, based on the scaffolding in previous parts, we can directly compute this using inclusion-exclusion again.

Let D be the event that there is at least one apple and grape flavor and E be the event that there is at least one orange and apple flavor. Then the event that Squidward is satisfied is $(D \cup E)^C$. Note that $\Pr[D] = \Pr[E] = p_1$ and $\Pr[D \cap E] = p_2$.

Then we have

$$\begin{aligned}
\Pr[(D \cup E)^C] &= 1 - \Pr[D \cup E] \\
&= 1 - (\Pr[D] + \Pr[E] - \Pr[D \cap E]) && \text{Inclusion-Exclusion} \\
&= 1 - 2p_1 + p_2 \\
(&= \frac{2279}{4606}) && \text{Unnecessary Computation}
\end{aligned}$$

4. Shrek Goes to Section [18 points]

Donkey is unfortunately falling behind in their assignments at this point of the quarter, and against his best judgment, he is strongly considering skipping section to catch up on studying. Donkey decides whether to go to section or not by asking Shrek, a friend in the same section.

- Shrek is a very reliable friend; he goes to section with probability s.
- If Shrek goes, Donkey is motivated and decides to go to section $8/9$ of the time.
- If Shrek doesn't go, Donkey is discouraged and skips section $4/7$ of the time.

Let D be the event that Donkey goes to section, and S be the event that Shrek goes to section.

(a) Which conditional probability or probability is referred to in the **last** bullet point?
Write the appropriate symbols below: [3 points]

$$\mathbb{P}\left(\rule{3cm}{0.4pt}\right)$$

Solution:

$\mathbb{P}\left(\overline{D} \mid \overline{S}\right)$

(b) What is the probability that Donkey goes to section?
Express your answer in terms of s. [5 points] **Solution:**

$$\begin{aligned}\mathbb{P}(D) &= \mathbb{P}(D|S)\mathbb{P}(S) + \mathbb{P}(D|\overline{S})\mathbb{P}(\overline{S}) && \text{Law of Total Probability} \\ &= \frac{8}{9}s + (1 - \frac{4}{7})(1 - s) \\ &= \frac{8}{9}s + \frac{3}{7}(1 - s)\end{aligned}$$

(c) If Donkey went to section, what is the probability that Shrek went to section?
Let d be the answer to part b. Express your answer in terms of d and s. [5 points] **Solution:**

$$\begin{aligned}\mathbb{P}(S \mid D) &= \frac{\mathbb{P}(D \mid S)\mathbb{P}(S)}{\mathbb{P}(D)} \\ &= \frac{\frac{8}{9}s}{d}\end{aligned}$$

(d) What is the probability that both Shrek and Donkey go to section?
You may express your answer in terms of previously defined variables. [5 points] **Solution:**

$$\mathbb{P}(S \cap D) = \mathbb{P}(S|D)\mathbb{P}(D) = \frac{8}{9}s$$

5. Random Variables [20 points]

A group of n friends is planning a book club. They have created a list of n books which they are considering reading. Each of the friends like exactly $n/2$ of the books (though which of the books could be different depending on which of the friends you ask). You may assume n is even so that $n/2$ is an integer. To form the club, each person will be given a set of 3 of the n books to read (each person gets three distinct books, but the same book can be assigned to any number of people). The set given to each person is uniformly random and independent of sets given to all others.

(a) Call a friend **very pleased** if they like **all** of the 3 books they read. What is the expected number of very pleased friends? [5 points] **Solution:**

Let Z_i be the indicator for "friend i is very pleased." Observe that $\mathbb{E}[Z_i] = \mathbb{P}[\text{friend } i \text{ is very pleased}] = \frac{\binom{n/2}{3}}{\binom{n}{3}}$

Let Z be the number of very pleased friends. Since $Z = \sum Z_i$, we have

$$\mathbb{E}[Z] = \mathbb{E}\left[\sum Z_i\right] = \sum_{i=1}^{n} \mathbb{E}[Z_i] = n \cdot \frac{\binom{n/2}{3}}{\binom{n}{3}}$$

(b) Once the books are assigned and read it's time to discuss! For each of the n books, every person who was assigned that book will meet to discuss. What is the expected number of books with **no** readers to discuss it? [5 points] **Solution:**

Let B_i be the indicator that book i has no readers, and B be the total number of such books. Observe that $\mathbb{E}[B_i] = \mathbb{P}[\text{book } i \text{ has no readers}] = \left(\frac{\binom{n-1}{3}}{\binom{n}{3}}\right)^n$ Note that this can be (but doesn't have to be!!) simplified to $\left(\frac{n-3}{n}\right)^n$. Also equivalent to $\left(1 - \frac{\binom{n-1}{2}}{\binom{n}{3}}\right)^n$.

Again, applying linearity we have the expected number of undiscussed books is $n \cdot \left(\frac{n-3}{n}\right)^n$

(c) Suppose that Edward is one of the friends. Define Y to be the number of books that Edward is assigned that he likes. Write the PMF for Y. Be sure to include all cases. [3 points]

Solution:

$$p_Y(z) = \begin{cases} \binom{n/2}{3}/\binom{n}{3} & \text{if } z = 0 \\ \binom{n/2}{1}\binom{n/2}{2}/\binom{n}{3} & \text{if } z = 1 \\ \binom{n/2}{2}\binom{n/2}{1}/\binom{n}{3} & \text{if } z = 2 \\ \binom{n/2}{3}/\binom{n}{3} & \text{if } z = 3 \\ 0 & \text{otherwise} \end{cases}$$

(d) One of the books is *Probability and Statistics with Applications to Computing* (by Alex Tsun). Let X be the number of friends who are given that book.

(i) What is the support of X? [3 points] **Solution:**

$\{0, 1, 2, \ldots, n\}$

(ii) what is Var(X)? [4 points] **Solution:**

This is a binomial random variable! Each trial is "does person i get assigned the book?" which happens with probability $\frac{\binom{n-1}{2}}{\binom{n}{3}} = \frac{3}{n}$. So the variance is $np(1-p) = n \cdot \frac{3}{n} \cdot \frac{n-3}{n}$ One can also approach this problem with indicators. Let X_i be the indicator for "friend i recieves the book" observe that $\sum X_i = X$ and that the X_i are independent. We can find the variance of each X_i as $p(1-p)$ (either by remembering the formula for indicators or calculating from scratch) and add the variances of the n copies (by independence) to get the same formula.

6. Grading Morale [1 point]

Robbie is playing on the softball team this quarter! Draw a picture, write a poem, or make a meme of a 312 topic that is interesting to you that somehow incorporates softball.