Homework 2: More Counting and Probability

For each problem, remember you must briefly explain/justify how you obtained your answer, as correct answers without an explanation will not receive full credit. Moreover, in the event of an incorrect answer, we can still try to give you partial credit based on the explanation you provide.

In general, your goal in an explanation is to write enough that a student from class who has attended lecture, but not read the problem yet, could understand your approach, verify your reasoning, and believe your answer is correct. While we do not usually need to see arithmetic, you must include enough work that in principle one could rederive your answer with only a scientific calculator.

Unless a problem states otherwise, you should leave your answer in terms of factorials, combinations, etc., for instance 26^7 or $26!/7!$ or $26 \cdot \binom{26}{7}$ are all good forms for final answers.

Submission: You must upload a **pdf** of your written solutions to Gradescope under "HW 2 [Written]". (Instructions as to how to upload your solutions to Gradescope are on the course web page.) The use of $\LaTeX$ is *highly recommended*. (Note that if you want to hand-write your solutions, you'll need to scan them. We will take off points for hand-written solutions that are difficult to read due to poor handwriting and neatness.)

You will submit the written problems as a PDF to Gradescope. Please put each numbered problem on its own page of the pdf (this will make selecting pages easier when you submit), and ensure that your pdfs are oriented correctly (e.g. not upside-down or sideways).

Due Date: This assignment is due Wednesday July 9th at 11:59 PM.

Academic Integrity: Please read the full academic integrity policy. If you work with others (and you should!), you must still write up your solution independently and name all of your collaborators in the separate question on gradescope.

1. Combinatorial Identities [16 points]

Prove each of the following identities using a *combinatorial argument* (i.e., an argument that counts two different ways); an algebraic solution will be marked substantially incorrect.

For the purposes of these problems, using commutativity of multiplication and addition (i.e. $ab = ba$, $a + b = b + a$), and distributivity/factoring ($a(b + c) = ab + ac$) are allowed as part of a combinatorial argument. Any other algebra facts (e.g. Pascal's Rule about combinations, the definition of combinations/permutations in terms of factorials, canceling numbers that appear in numerators and denominators) would make it an algebraic solution, not a combinatorial one.

(a) $\displaystyle\sum_{i=0}^{x} \binom{x}{i}\binom{y}{i} = \binom{x+y}{x}$. You may assume that $y \geq x \geq 0$.
Hint: Start with the right hand side. Imagine you are choosing a team of x elder Jedi, from a group consisting of the current x elders and y new applicants.

(b) $\displaystyle\sum_{i=x}^{y} \binom{y}{i}\binom{i}{x} = \binom{y}{x}2^{y-x}$. Assume that $y \geq x \geq 0$.
Hint: Think about choosing a Jedi council of varying size from y Jedi, with a executive sub-council of fixed size.

2. Cabbage merchants [8 points]

Competition is fierce for the 25 cabbage merchants in Ba Sing Se. The vendors are laid out in a 5x5 grid with each merchant occupying a single unique square. City law dictates that the merchants **must** move to an adjacent square in the following morning (either vertically, horizontally, but not diagonally). If two or more merchants end up on the same square, they will begin a honorary cabbage duel. Argue that the probability that a cabbage duel occurs is 1 on the following morning.

Hint: Draw and represent the 5x5 grid as a chessboard.

3. Name, Major, Something Fun You Did Over Break [10 points]

Your TA has assigned your class a very fun and totally original icebreaker. In a large classroom, there won't be time for every student to introduce themselves to every other student, though some students may introduce themselves with multiple others.

Suppose you have a group of n students. Assign each student a "tally", which is the number of other students they did the icebreaker with.

Use the pigeonhole principle to show the following claim:

In a group of n students, there are at least two with the same tally score.

You can assume that the icebreaker is symmetric. If A introduces themselves to B, then so does B to A. Furthermore one cannot do the ice breaker with themselves. When using the pigeonhole principle, be sure to mention what the pigeons are and pigeonholes are.

Hint: You may want to break into cases; think about what happens if a student has a score of 0.

4. Hogwarts Logistics [12 points]

(a) We have (20) distinguishable wizards and 55 (distinguishable) dorm rooms. How many ways are there to assign the (distinguishable) wizards to the (distinct) dorm rooms? Note: Any number of wizards can go into any of the 55 dorm rooms.

(b) We have 35 identical (indistinguishable) wands. How many different ways are there to distribute the wands among 15 (distinguishable) wizards? (Any number of wands can be given to any of the wizards.)

(c) We have 56 identical (indistinguishable) wands. How many different ways are there to distribute the wands among 12 (distinguishable) wizard, if each wizard must receive at least three wands (they break wands frequently and need backups)?

(d) We want to choose a team of six wizards, which may include wizards in any of the four houses (Gryffindor, Ravenclaw, Hufflepuff, and Slytherin). How many ways are there to choose a team that includes at least one Gryffindor, **or** at least one Hufflepuff?

5. Sample Spaces and Probabilities [online]

Do the gradescope question asking for sizes of sample spaces and probability computations. We will not be able to award partial credit, so please enter your submissions carefully.

6. Miscounting [14 points]

Consider the question: How many **7-card** poker hands (order doesn't matter) are there that contain at least three pairs (pairs means two cards of the same value). For example, this would be a valid hand: ace of hearts, ace of diamonds, king of spades, king of diamonds, 7 of spades, 7 of hearts, and queen of clubs. (Note that a hand consisting of three aces, two kings, and two queens counts but a hand consisting of four aces, two kings, and one queen does not. You must have three distinct values with at least two cards of that value each)

Here is how we might compute this:

> To compute the number of hands, apply the product rule using the following sequential process. First pick three ranks that have a pair (e.g. ace, king, 7 in the example above). For the lowest rank of these, pick the suits of the two cards. Then for the middle rank of these, pick the suits of the two cards, then for the highest rank of these, pick the suits of the two cards. Then out of the remaining $52 - 6 = 46$ cards, pick one. Therefore there are
>
> $$\binom{13}{3} \cdot \binom{4}{2} \cdot \binom{4}{2} \cdot \binom{4}{2} \cdot \binom{46}{1}$$
>
> hands.

In this problem, you will find what is wrong with this solution.

(a) Is there overcounting in the solution? That is, is there a hand that can be produced by multiple outcomes of the sequential process? If there is, give one concrete example of such a hand and two outcomes of the process that produce it. If there is not, briefly (1-2 sentences) explain why there isn't.

(b) Is there undercounting in the problem? That is, is there a hand that cannot be produced by any outcomes of the sequential process? If there is, give one concrete example of such a hand and briefly explain why no outcome produces it. If there is no such hand, briefly explain why all hands are produced at least once.

(c) Correct the calculation – in this part you should produce a correct overall formula by subtracting/dividing out any errors that would fit in (a) and adding/multiplying in any errors that would fit in (b).

(d) Find the answer differently – take a different approach to counting this problem (e.g. use a different sequential process). Verify that you get the same number (via a different formula) than the last part.

7. Classic [Extra Credit]

You will submit this question to the separate gradescope box for "homework 2 extra credit."

The goals of this problem are to

- Practice induction (remember induction? It's back!) so that you don't totally forget it before you write another inductive proof in 421.
- Realize that while induction works for proving combinatorial identities, it usually leads to longer proofs than other methods.

(a) Use Pascal's Rule to prove that for any $n \geq 0$

$$\sum_{k=0}^{n} \binom{n}{k} + \sum_{k=0}^{n} \binom{n}{k} = \sum_{k=0}^{n+1} \binom{n+1}{k}$$

(you do not need to use induction for this part)

(b) Now, prove via induction that $\sum_{k=0}^{n} \binom{n}{k} = 2^n$ holds for all natural numbers n. You may use part (a) in your inductive step.

(c) We've seen in lecture that this theorem can be proven very quickly via the binomial theorem. There is also a combinatorial proof (counting subsets of a set of size n, on the left hand side we consider the subsets of size k); of the three versions of this proof (binomial theorem, induction, combinatorial) which do you prefer? Why? (Write 1-3 sentences; there are no right or wrong answers).

8. Feedback + Collaboration [1 point]

Answer these questions on the separate Gradescope box for this question.

Please keep track of how much time you spend on this homework and answer the following questions. This can help us calibrate future assignments and future iterations of the course, and can help you identify which areas are most challenging for you.

- Which students did you collaborate with for this homework?
- Is the work that you are submitting your own and does not violate the academic integrity policy outlined in the syllabus?
- How many hours did you spend working on this assignment (excluding any extra credit questions, if applicable)? Report your estimate to the nearest hour.
- Which problem did you spend the most time on?
- Any other feedback for us?