Section 4: Solutions

Review of Main Concepts

- **Random Variable (rv)**: A numeric function $X : \Omega \to \mathbb{R}$ of the outcome.
- **Range/Support:** The support/range of a random variable X, denoted Ω_X, is the set of all possible values that X can take on.
- **Discrete Random Variable (drv)**: A random variable taking on a countable (either finite or countably infinite) number of possible values.
- **Probability Mass Function (pmf) for a discrete random variable X**: a function $p_X : \Omega_X \to [0, 1]$ with $p_X(x) = \mathbb{P}(X = x)$ that maps possible values of a discrete random variable to the probability of that value happening, such that $\sum_x p_X(x) = 1$.
- **Cumulative Distribution Function (CDF) for a random variable X**: a function $F_X : R \to R$ with $F_X(x) = \mathbb{P}(X \leq x)$
- **Expectation (expected value, mean, or average)**: The expectation of a discrete random variable is defined to be $\mathbb{E}[X] = \sum_x x p_X(x) = \sum_x x\mathbb{P}(X = x)$. The expectation of a function of a discrete random variable $g(X)$ is $\mathbb{E}[g(X)] = \sum_x g(x) p_X(x)$.
- **Linearity of Expectation**: Let X and Y be random variables, and $a, b, c \in \mathbb{R}$. Then, $\mathbb{E}[aX + bY + c] = a\mathbb{E}[X] + b\mathbb{E}[Y] + c$. Also, for any random variables $X_1, \ldots, X_n$,

$$\mathbb{E}[X_1 + X_2 + \ldots + X_n] = \mathbb{E}[X_1] + \mathbb{E}[X_2] + \ldots + \mathbb{E}[X_n].$$

- **Variance**: Let X be a random variable and $\mu = \mathbb{E}[X]$. The variance of X is defined to be $Var(X) = \mathbb{E}[(X-\mu)^2]$. Notice that since this is an expectation of a non-negative random variable ($(X - \mu)^2$), variance is always non-negative. With some algebra, we can simplify this to $Var(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$.
- **Standard Deviation**: Let X be a random variable. We define the standard deviation of X to be the square root of the variance, and denote it $\sigma = \sqrt{Var(X)}$.
- **Property of Variance**: Let $a, b \in \mathbb{R}$ and let X be a random variable. Then, $Var(aX + b) = a^2 Var(X)$.
- **Independence**: Random variables X and Y are independent iff

$$\forall x \forall y, \quad \Pr(X = x \cap Y = y) = \Pr(X = x)\Pr(Y = y)$$

  In this case, we have $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$ (the converse is not necessarily true).
- **i.i.d. (independent and identically distributed)**: Random variables $X_1, \ldots, X_n$ are i.i.d. (or iid) iff they are independent and have the same probability mass function.
- **Variance of Independent Variables**: If X is independent of Y, $\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y)$. This depends on independence, whereas linearity of expectation always holds. Note that this combined with the above shows that $\forall a, b, c \in \mathbb{R}$ and if X is independent of Y, $Var(aX + bY + c) = a^2 Var(X) + b^2 Var(Y)$.

1. Bayes' Rule Practice

Suppose that you can commute to campus via bike or public transit. If it's raining, you'll bike to campus 20% of the time, and if it's not raining, you'll bike to campus 60% of the time. You know that it rains 7 days out of 10 during Autumn quarter. If you biked to campus today, what is the probability that it was raining?

Solution:

Let B be the event that you biked to campus, and R be the event that it rained. We want to calculate $P(R|B)$. By Bayes' Rule, we have

$$P(R|B) = \frac{P(B|R) \cdot P(R)}{P(B)}$$

To solve for the denominator, we use the law of total probability:

$$P(B) = P(R) \cdot P(B|R) + P(R^C) \cdot P(B|R^C)$$

So, we have

$$P(R|B) = \frac{P(B|R) \cdot P(R)}{P(R) \cdot P(B|R) + P(R^C) \cdot P(B|R^C)}$$

We are given in the problem the probability of biking given raining or not raining, and the probability of it raining. So, we know the following:

$$P(B|R) = 0.2$$
$$P(B|R^C) = 0.6$$
$$P(R) = 0.7$$

We know that

$$\begin{aligned} P(R^C) &= 1 - P(R) && (1)\\ &= 1 - 0.7 && (2)\\ &= 0.3 && (3) \end{aligned}$$

Plugging in these values, we get that

$$\begin{aligned} P(R|B) &= \frac{P(B|R) \cdot P(R)}{P(R) \cdot P(B|R) + P(R^C) \cdot P(B|R^C)} && (4)\\ &= \frac{0.2 \cdot 0.7}{0.2 \cdot 0.7 + 0.6 \cdot 0.3} && (5)\\ &= \frac{0.14}{0.14 + 0.18} && (6)\\ &= 0.4375 && (7) \end{aligned}$$

2. Content Review

(a) True or false: the range of a random variable X is the set of probabilities corresponding to the possible values X can take on. **Solution:**

False. The range (or *support*) of a random variable is the set of all possible values it can take on.

(b) What is the relationship between standard deviation and variance of a random variable X?

- ◯ $\sigma = (V(X))^2$
- ◯ $\sigma = Var(X^2)$
- ◯ $Var(X) = \sigma^2$

Solution:

$Var(X) = \sigma^2$ (or $\sigma = \sqrt{Var(X)}$ in the above review)

(c) Let X be the random variable representing the outcome of taking the sum of a 3-dice roll of 6-sided dice. Which function would you use to determine the probability that $X = 7$?

◯ CDF (cumulative distribution function)

◯ PMF (probability mass function)

Solution:

PMF. We use the PMF when we want to find the probability of a specific value of a random variable occurring.

(d) Let X be the random variable representing the outcome of taking the sum of a 3-dice roll of 6-sided dice. Which function would you use to determine the probability that $X \leq 7$?

◯ CDF (cumulative distribution function)

◯ PMF (probability mass function)

Solution:

CDF. The CDF gives us exactly $\mathbb{P}(X \leq x)$.

(e) A random variable X has the PMF

$$p_X(x) = \begin{cases} 1/4 & x = -1 \\ 1/4 & x = 0 \\ 1/2 & x = 2 \\ 0 & \text{otherwise} \end{cases}$$

What is $\mathbb{E}[X]$?

◯ -1/4

◯ 3/4

◯ 1

◯ 2

Solution:

3/4.

$$\mathbb{E}[X] = \sum_{x \in \Omega_X} x p_X(x) = -1 \cdot 1/4 + 0 \cdot 1/4 + 2 \cdot 1/2 = 3/4.$$

(f) A random variable X has the PMF

$$p_X(x) = \begin{cases} 1/4 & x = -1 \\ 1/4 & x = 0 \\ 1/2 & x = 2 \\ 0 & \text{otherwise} \end{cases}$$

What is $Var[X]$?

◯ $3/4$

◯ 1

◯ $((1/4) + 2) - ((\frac{3}{4})^2) = 27/16$

◯ $((1/4) + 2) + ((\frac{3}{4})^2) = 45/16$

Solution:

27/16.

$$\begin{aligned} Var[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2 &= \sum_{x\in\Omega_X} x^2 p_X(x) - \sum_{x\in\Omega_X} x p_X(x) \\ &= ((-1)^2 \cdot 1/4 + 0^2 \cdot 1/4 + 2^2 \cdot 1/2) - ((3/4)^2) \\ &= 27/16 \end{aligned}$$

3. Identify that Range!

Identify the support/range Ω_X of the random variable X, if X is...

(a) The sum of two rolls of a six-sided die. **Solution:**

X takes on every integer value between the min sum 2, and the max sum 12.
$\Omega_X = \{2, 3, ..., 12\}$

(b) The number of lottery tickets I buy until I win it. **Solution:**

X takes on all positive integer values (I may never win the lottery).
$\Omega_X = \{1, 2, ...\}$

(c) The number of heads in n flips of a coin with $0 < \mathbb{P}(\text{head}) < 1$. **Solution:**

X takes on every integer value between the min number of heads 0, and the max n.
$\Omega_X = \{0, 1, ..., n\}$

(d) The number of heads in n flips of a coin with $\mathbb{P}(\text{head}) = 1$. **Solution:**

Since $\mathbb{P}(head) = 1$, we are guaranteed to get n heads in n flips.
$\Omega_X = \{n\}$

(e) The number of whole minutes I wait at the bus stop for the next bus. **Solution:**

The number of whole minutes is discrete and will take on values between the minimum waiting time (0, the bus is here), and the maximum waiting time (∞, the bus never gets here).
$\Omega_X = \{0, 1, ...\}$

4. 3-sided Die

Let the random variable X be the sum of two independent rolls of a fair 3-sided die. (If you are having trouble imagining what that looks like, you can use a 6-sided die and change the numbers on 3 of its faces.)

(a) What is the probability mass function of X? **Solution:**

First let us define the range of X. A three sided-die can take on values $1, 2, 3$. Since X is the sum of two rolls, the range of X is $\Omega_X = \{2, 3, 4, 5, 6\}$.

We can then define the pmf of X. To that end, we must define two random variables R_1, R_2 with R_1 being the roll of the first die, and R_2 being the roll of the second die. Then, $X = R_1 + R_2$. Note that $\Omega_{R1} = \Omega_{R2} = \{1, 2, 3\}$. With that in mind we can find the pmf of X:

$$\begin{aligned}
p_X(k) = \text{Pr}(X = k) &= \sum_{i \in \Omega_{R1}} \text{Pr}(R_1 = i, R_2 = k - i) \\
&= \sum_{i \in \Omega_{R1}} \text{Pr}(R_1 = i) \cdot \text{Pr}(R_2 = k - i) \qquad \text{(By independence of the rolls)} \\
&= \sum_{i \in \Omega_{R1}} \frac{1}{3} \cdot p_{R2}(k - i) \\
&= \frac{1}{3} \left(p_{R2}(k-1) + p_{R2}(k-2) + p_{R2}(k-3) \right)
\end{aligned}$$

At this point, we can evaluate the pmf of X for each value in the range of X, noting that $p_{R2}(k - i) = 0$ if

$k - i \notin \Omega_{R2}$, $1/3$ otherwise. We get:

$$p_X(k) = \begin{cases} 1/9 & k = 2 \\ 2/9 & k = 3 \\ 3/9 & k = 4 \\ 2/9 & k = 5 \\ 1/9 & k = 6 \\ 0 & \text{otherwise} \end{cases}$$

One could also list out the possible values of the first two rolls and use a table to find the marginal pmf of X by summing up the entries of each row for each $k \in \Omega_X$.

(b) Find $\mathbb{E}[X]$ directly from the definition of expectation. **Solution:**

$$\mathbb{E}[X] = \sum_{k=2}^{6} k p_X(k) = 2 \cdot \frac{1}{9} + 3 \cdot \frac{2}{9} + 4 \cdot \frac{3}{9} + 5 \cdot \frac{2}{9} + 6 \cdot \frac{1}{9} = \boxed{4}$$

(c) Find $\mathbb{E}[X]$ again, but this time using linearity of expectation. **Solution:**

Let R_1 be the roll of the first die, and R_2 the roll of the second. Then, $X = R_1 + R_2$.
By linearity of expectation, we get:

$$\mathbb{E}[X] = \mathbb{E}[R_1 + R_2] = \mathbb{E}[R_1] + \mathbb{E}[R_2]$$

We compute:

$$\mathbb{E}[R_1] = \sum_{i \in \Omega_{R1}} i \cdot \Pr(R_1 = i) = \sum_{i \in \Omega_{R1}} i \cdot \frac{1}{3} = \frac{1}{3}(1 + 2 + 3) = 2$$

Similarly, $E[R_2] = 2$, since the rolls are independent.

Plugging into our expression for the expectation of X gives us:

$$E[X] = 2 + 2 = \boxed{4}$$

(d) What is $Var(X)$? **Solution:**

We know from the definition of variance that

$$Var(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

We can compute the $\mathbb{E}[X^2]$ term as follows:

$$\mathbb{E}[X^2] = \sum_{x=2}^{6} x^2 p_X(x) = \frac{2^2 \cdot 1 + 3^2 \cdot 2 + 4^2 \cdot 3 + 5^2 \cdot 2 + 6^2 \cdot 1}{9} = \frac{52}{3}$$

Plugging this into our variance equation gives us

$$Var(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \frac{52}{3} - 4^2 = \boxed{\frac{4}{3}}$$

5. Kit Kats Again

Suppose we have N candies in a jar, K of which are kit kats. Suppose we draw (without replacement) until we have (exactly) k kit kats, $k \leq K \leq N$. Let X be the number of draws until the k^{th} kit kat. What is Ω_X, the range of X? What is $p_X(n) = \Pr(X = n)$? (We say X is a "negative hypergeometric" random variable). **Solution:**

Consider all N candies to be arranged randomly in a row, We will treat the first n candies to be the "chosen" candies. We know that the nth candy is a kitkat (the kth kitkat) to be chosen. This nth candy divides the row into 2 sections. On the left, are the $n-1$ candies that were chosen (all candies chosen except the last kitkat). On the right, are the candies remaining in the jar. So the first term in the numerator is choosing the $k-1$ spots where kitkats will be chosen from the $n-1$ spots. The second term is choosing the $K-k$ spots for the kitkats that remain in the jar among the remaining $N-n$ candies. The denominator is the total number of possible ways to place the K kitkats among N candies.

$$\Omega_X = \{k, k+1, \dots N - K + k\}$$

$$p_X(n) = \Pr(X = n) = \begin{cases} \dfrac{\binom{n-1}{k-1}\binom{N-n}{K-k}}{\binom{N}{K}} & k \in \Omega_X \\ 0 & \text{otherwise} \end{cases}$$

6. Hungry Washing Machine

You have 10 pairs of socks (so 20 socks in total), with each pair being a different color. You put them in the washing machine, but the washing machine eats 4 of the socks chosen at random. Every subset of 4 socks is equally probable to be the subset that gets eaten. Let X be the number of complete pairs of socks that you have left.

(a) What is the range of X, Ω_X (the set of possible values it can take on)? What is the probability mass function of X? **Solution:**

The washing machine eats 4 socks every time. It can either eat a single sock from 4 pairs of socks, leaving us with 6 complete pairs, or a single sock from 2 pairs and a matching pair, leaving us with 7 complete pairs, or 2 pairs of matching socks, leaving us with 8 complete pairs.

$$\Omega_X = \{6, 7, 8\}$$

We are dealing with a sample space with equally likely outcomes. As such, we can compute use the formula $P(E) = \frac{|E|}{|\Omega|}$. We know that $|\Omega| = \binom{20}{4}$ because the washing machine picks a set of 4 socks out of 20

possible socks.

To define the pmf of X, we consider each value in the range of X.

For $k = 6$, we first pick 4 out of 10 pairs of socks from which we will eat a single sock ($\binom{10}{4}$ ways), and for each of these 4 pairs we have two socks to pick from ($\binom{2}{1}^4$ ways). Using the product rule, we get $|X = 6| = \binom{10}{4}2^4$.

For $k = 7$, we first pick 1 out of 10 pairs of socks to eat in its entirety ($\binom{10}{1}$ ways), and then 2 out of the 9 remaining pairs from which we will eat a single sock ($\binom{9}{2}$ ways), and for each of these 2 pairs we have two socks to pick from ($\binom{2}{1}^2$ ways). Using the product rule, we get $|X = 7| = 10\binom{9}{2}2^2$.

For $k = 8$, we pick 2 out of 10 pairs of socks to eat ($\binom{10}{2}$ ways). We get $|X = 8| = \binom{10}{2}$.

$$p_X(k) = \begin{cases} \frac{\binom{10}{4}2^4}{\binom{20}{4}} & k = 6 \\ \frac{10\binom{9}{2}2^2}{\binom{20}{4}} & k = 7 \\ \frac{\binom{10}{2}}{\binom{20}{4}} & k = 8 \\ 0 & \text{otherwise} \end{cases}$$

(b) Find $\mathbb{E}[X]$ from the definition of expectation. **Solution:**

$$\mathbb{E}[X] = \sum_{k \in \Omega_X} k \cdot p_X(k) = 6 \cdot \frac{\binom{10}{4}2^4}{\binom{20}{4}} + 7 \cdot \frac{10\binom{9}{2}2^2}{\binom{20}{4}} + 8 \cdot \frac{\binom{10}{2}}{\binom{20}{4}} = \boxed{\frac{120}{19}}$$

(c) Find $\mathbb{E}[X]$ using linearity of expectation. (see this video for a walkthrough of this problem!) **Solution:**

For $i \in [10]$, let X_i be 1 if pair i survived, and 0 otherwise. Then, $X = \sum_{i=1}^{10} X_i$. But $\mathbb{E}[X_i] = 1 \cdot \Pr(X_i = 1) + 0 \cdot \Pr(X_i = 0) = \Pr(X_i = 1) = \frac{\binom{18}{4}}{\binom{20}{4}}$, where the numerator indicates the number of ways of choosing 4 out the 18 remaining socks (we spare our chosen pair i). Hence,

$$\mathbb{E}[X] = \mathbb{E}[\sum_{i=1}^{10} X_i] = \sum_{i=1}^{10} \mathbb{E}[X_i] = \sum_{i=1}^{10} \frac{\binom{18}{4}}{\binom{20}{4}} = 10 \frac{\binom{18}{4}}{\binom{20}{4}} = \boxed{\frac{120}{19}}$$

(d) Which way was easier? Doing both (a) and (b), or just (c)? **Solution:**

Part (c) is was probably much easier. In this problem, you may have found part (a) and (b) easier, because there were only 3 possible values in the range of X. However, in general computing the probability mass function of complicated random variables (ones with hundreds of elements in their range) can be very difficult. Often it is much easier to use linearity of expectation and compute the probability mass function of simpler random variables.

7. Hat Check

At a reception, n people give their hats to a hat-check person. When they leave, the hat-check person gives each of them a hat chosen at random from the hats that remain. What is the expected number of people who get their own hats back? (Notice that the hats returned to two people are not independent events: if a certain hat is returned to one person, it cannot also be returned to the other person.)

Note: See this video for a walkthrough of this problem!

Solution:

Let X be the number of people who get their hats back. For $i \in [n]$, let X_i be 1 if person i gets their hat back, and 0 otherwise. Then, $\mathbb{E}[X_i] = \Pr(X_i = 1) = \frac{|E|}{|\Omega|}$. The sample space is all possible distributions of hats among the n people, and the event of interest E is the subset of the sample space where person i has their own hat. There are $n!$ ways to distribute the n hats among the n people. This is because the first person might have gotten 1 out of n possible hats; for each hat the first person got, the second person could get $n-1$ possible hats; and so on. The number of ways person i can get their hat back is $(n-1)!$. This is because we are essentially removing person i and hat i from the pool of people/hats, and counting the permutations of the $n-1$ remaining people.

Thus, $\Pr(X_i = 1) = \frac{(n-1)!}{n!} = \frac{1}{n}$. Since $X = \sum_{i=1}^{n} X_i$, we have

$$\mathbb{E}[X] = \mathbb{E}[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} \mathbb{E}[X_i] = \sum_{i=1}^{n} \frac{1}{n} = n \cdot \frac{1}{n} = 1$$

8. Frogger

A frog starts on a 1-dimensional number line at 0. At each second, independently, the frog takes a unit step right with probability p_1, to the left with probability p_2, and doesn't move with probability p_3, where $p_1 + p_2 + p_3 = 1$. After 2 seconds, let X be the location of the frog.

(a) Find $p_X(k)$, the probability mass function for X. **Solution:**

Let L be a left step, R be a right step, and N be no step.

The range of X is $\{-2, -1, 0, 1, 2\}$. We can compute $p_X(-2) = \Pr(X = -2) = \Pr(LL) = p_2^2$, $p_X(-1) = \Pr(X = -1) = \Pr(LN \cup NL) = 2p_2p_3$, and $p_X(0) = \Pr(X = 0) = \Pr(NN \cup LR \cup RL) = p_3^2 + 2p_1p_2$. Similarly for $p_X(1)$ and $p_X(2)$.

$$p_X(k) = \begin{cases} p_2^2 & k = -2 \\ 2p_2p_3 & k = -1 \\ p_3^2 + 2p_1p_2 & k = 0 \\ 2p_1p_3 & k = 1 \\ p_1^2 & k = 2 \\ 0 & \text{otherwise} \end{cases}$$

(b) Compute $\mathbb{E}[X]$ from the definition.

Solution:

$$\mathbb{E}[X] = (-2)(p_2^2) + (-1)(2p_2p_3) + (0)(p_3^2 + 2p_1p_2) + (1)(2p_1p_3) + (2)(p_1^2) = 2(p_1 - p_2)$$

(c) Compute $\mathbb{E}[X]$ again, but using linearity of expectation. **Solution:**

Let Y be the amount you moved on the first step (either $-1, 0, 1$), and Z the amount you moved on the second step. Then, $\mathbb{E}[Y] = \mathbb{E}[Z] = (1)(p_1) + (0)(p_3) + (-1)(p_2) = p_1 - p_2$.

Then $X = Y + Z$ and $\mathbb{E}[X] = \mathbb{E}[Y + Z] = \mathbb{E}[Y] + \mathbb{E}[Z] = 2(p_1 - p_2)$

9. Balls in Bins

Let X be the number of bins that remain empty when m balls are distributed into n bins randomly and independently. For each ball, each bin has an equal probability of being chosen. (Notice that two bins being empty are not independent events: if one bin is empty, that decreases the probability that the second bin will also be empty. This is particularly obvious when $n = 2$ and $m > 0$.) Find $\mathbb{E}[X]$. **Solution:**

For $i \in [n]$, let X_i be 1 if bin i is empty, and 0 otherwise. Then, $X = \sum_{i=1}^{n} X_i$. We first compute $\mathbb{E}[X_i] =$

$1 \cdot \Pr(X_i = 1) + 0 \cdot \Pr(X_i = 0) = \Pr(X_i = 1) = (\frac{n-1}{n})^m$. Indeed, we are assuming multiple balls can go in the same bin. As such, when computing $P(X_i = 1)$, given that bin i is empty, we remove it from the pool of possible bins to pick from, leaving us with $n - 1$ bins out of a total of n bins in which we can place balls. Since we are distributing m balls over the n bins, the event that bin i remains empty occurs with probability $\left(\frac{n-1}{n}\right)^m$. Hence, by linearity of expectation:

$$\mathbb{E}[X] = \mathbb{E}[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} \mathbb{E}[X_i] = n \cdot \left(\frac{n-1}{n}\right)^m$$

10. Fair Game?

You flip a fair coin independently and count the number of flips until the first tail, including that tail flip in the count. If the count is n, you receive 2^n dollars. What is the expected amount you will receive? How much would you be willing to pay at the start to play this game? **Solution:**

The expected amount is ∞. Let N be the number of flips until the first tail, so $p_N(n) = \frac{1}{2^n}$ for $n \in \mathbb{N}$ (independent flips of a fair coin; $\mathbb{N}$ is the range of N and refers to the set of natural numbers). We have $\mathbb{E}[2^N] = \sum_{n=1}^{\infty} 2^n \frac{1}{2^n} = \sum_{n=1}^{\infty} 1 = \infty$. In theory, you should be willing to pay any finite amount of money to play this game, but I admit I would be nervous to pay a lot. For instance, if you pay \$1000, you will lose money unless the first 9 flips are all heads. With high probability you will lose money, and with low probability you will win a lot of money.

11. Symmetric Difference

Suppose A and B are random, independent (possibly empty) subsets of $\{1, 2, \ldots, n\}$, where each subset is equally likely to be chosen as A or B. Consider $A\Delta B = (A \cap B^C) \cup (B \cap A^C) = (A \cup B) \cap (A^C \cup B^C)$, i.e., the set containing elements that are in exactly one of A and B. Let X be the random variable that is the size of $A\Delta B$. What is $\mathbb{E}[X]$? **Solution:**

For $i = 1, 2, \ldots, n$, let X_i be the indicator of whether $i \in A\Delta B$. Then $\mathbb{E}[X_i] = \Pr(X_i = 1) = \frac{1}{2}$ (every subset of $1, 2, .., n$ either contains i or it does not), and $X = \sum_{i=1}^{n} X_i$, so

$$\mathbb{E}[X] = \mathbb{E}[\sum_{i=1}^{n} X_i] = \frac{n}{2}$$

.

12. Practice

(a) Let X be a random variable with $p_X(k) = ck$ for $k \in \{1, \ldots, 5\} = \Omega_X$, and 0 otherwise. Find the value of c that makes X follow a valid probability distribution and compute its mean and variance ($E[X]$ and $Var(X)$).

(b) Let X be *any* random variable with mean $E[X] = \mu$ and variance $Var(X) = \sigma^2$. Find the mean and variance of $Z = \dfrac{X - \mu}{\sigma}$. (When you're done, you'll see why we call this a "standardized" version of X!)

(c) Let X, Y be independent random variables. Find the mean and variance of $X - 3Y - 5$ in terms of $E[X], E[Y], Var(X)$, and $Var(Y)$.

(d) Let $X_1, \ldots, X_n$ be independent and identically distributed (iid) random variables each with mean μ and variance σ^2. The sample mean is $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$. Find the mean and variance of $\bar{X}$. If you use the independence assumption anywhere, **explicitly label** at which step(s) it is necessary for your equalities to be true.

Solution:

(a) For X to follow a valid probability distribution, we must have $\sum_{k\in\Omega_X} p_X(k) = 1$. We can solve for c so that the equality holds. We know:

$$\sum_{k\in\Omega_X} p_X(k) = \sum_{k\in\Omega_X} ck = c \sum_{k\in\Omega_X} k) = c \cdot (1+2+3+4+5) = 15c$$

So for the normalization of the pmf of X to hold, we must choose $c = 1/15$.
We can now use the definition of expectation:

$$E[X] = 1\cdot\frac{1}{15} + 2\cdot\frac{2}{15} + 3\cdot\frac{3}{15} + 4\cdot\frac{4}{15} + 5\cdot\frac{5}{15} = 55/15 \approx \boxed{3.667}$$

And compute $E[X]$ as follows:

$$E[X^2] = 1^2\cdot\frac{1}{15} + 2^2\cdot\frac{2}{15} + 3^2\cdot\frac{3}{15} + 4^2\cdot\frac{4}{15} + 5^2\cdot\frac{5}{15} = 225/15 = \boxed{15}$$

And the variance of X:

$$Var(X) = E[X^2] - E^2[X] = 15 - (55/15)^2 = \frac{15^3 - 55^2}{15} = \frac{350}{225} = \frac{14}{9} \approx \boxed{1.556}$$

(b) We know that $E[aX] = a \cdot E[X]$ for some constant a, and that $E[X+b] = E[X] + b$ for some constant b. As such, we can compute the expectation of the standardized version of X, knowing that $E[X] = \mu$:

$$E[Z] = E\left[\frac{X-\mu}{\sigma}\right] = \frac{1}{\sigma}\left(E[X-\mu]\right) = \frac{1}{\sigma}(E[X]-\mu) = \boxed{0}$$

For the variance, we know that $Var(aX+b) = a^2 Var(X)$. With that in mind, knowing that $Var(X) = \sigma^2$, we can write:

$$Var(Z) = Var\left(\frac{X-\mu}{\sigma}\right) = \frac{1}{\sigma^2}Var(X) = \boxed{1}$$

(c) Using the linearity of expectation, we can write:

$$E[X - 3Y - 5] = E[X] - 3E[Y] - 5$$

We also know that the variance of a sum of independent random variables A and B is the sum of their variances, so that $Var(A+B) = Var(A) + Var(B)$. In our case, we have $A = X$, and $B = -3Y$. We get:

$$Var(X - 3Y - 5) = Var(X) + Var(-3Y) = Var(X) + 9Var(Y)$$

(d) Using linearity of expectation,

$$E[\overline{X}] = E\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \frac{1}{n}\sum_{i=1}^{n} E[X_i] = \frac{1}{n}n\mu = \mu$$

$$Var(\overline{X}) = Var\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2}\sum_{i=1}^{n} Var(X_i) = \frac{1}{n^2} n\sigma^2 = \frac{\sigma^2}{n}$$

In the calculation for the variance, we used the independence of the X_i's.

13. Coin Flipping

Suppose we have a coin with probability p of heads. Suppose we flip this coin until we flip a head for the first time. Let X be the number of times we flip the coin *up to and including* the first head. What is $\Pr(X = k)$, for $k = 1, 2, \ldots$? Verify that $\sum_{k=1}^{\infty} \Pr(X = k) = 1$, as it should. (You may use the fact that $\sum_{j=0}^{\infty} a^j = \frac{1}{1-a}$ for $|a| < 1$). **Solution:**

$$\Pr(X = k) = (1-p)^{k-1}p$$

If the k^{th} flip is our first head, the first $k-1$ must be tails (each with probability $(1-p)$, and the k^{th} flip must be a head with probability p.

$$\sum_{k=1}^{\infty} \Pr(X = k) = \sum_{k=1}^{\infty} (1-p)^{k-1}p = p\sum_{j=0}^{\infty}(1-p)^j = \frac{p}{1-(1-p)} = 1$$

(We set $j = k-1$ so our summation's lower bound $k = 1$ turned into $j = 0$).

14. More Coin Flipping ...

Suppose we have a coin with probability p of heads. Suppose we flip this coin n times independently. Let X be the number of heads that we observe. What is $\Pr(X = k)$, for $k = 0, \ldots n$? Verify that $\sum_{k=0}^{n} \Pr(X = k) = 1$, as it should. **Solution:**

$$\Pr(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

For a given sequence with exactly k heads, the probability of that sequence is $p^k(1-p)^{n-k}$. However, there are $\binom{n}{k}$ such sequences, so the probability of exactly k heads is $\binom{n}{k}p^k(1-p)^{n-k}$.

$$\sum_{k=0}^{n} \Pr(X = k) = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = (p + (1-p))^n = 1$$

The middle equality uses the Binomial Theorem.