CSE 312 Section 4

Random Variables and Expectation

Administrivia

Announcements & Reminders

- HW2
  - Grades released on gradescope – check your submission to read comments
  - Regrade requests open ~24 hours after grades are released and close after a week

- HW3
  - Was Due Yesterday, 10/15 @11:59pm
    - Late deadline Saturday 10/18 @ 11:59pm
  - # late days used for HW3 = Max(late days for written, late days for coding)
    - If you submit one day late for homework but two days late for coding, it counts for two late days total

- HW4
  - Released on the course website
  - Due Wednesday 10/22 @ 11:59pm

Problem 0 – Bayes’ Rule Practice

0 - Bayes' Rule Practice

Suppose that you can commute to campus via bike or public transit. If it's raining, you'll bike to campus 20% of the time, and if it's not raining, you'll bike to campus 60% of the time. You know that it rains 7 days out of 10 during Autumn quarter. If you biked to campus today, what is the probability that it was raining?

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Let B be the event that you biked to campus, and R be the event that it rained. We want to calculate $P(R|B)$.

Work on finding the probability with the people around you!

0 - Bayes' Rule Practice

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Let B be the event that you biked to campus, and R be the event that it rained. We want to calculate $P(R|B)$. By Bayes' Rule, we have

$$P(R|B) = \frac{P(B|R) \cdot P(R)}{P(B)}$$

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$$P(R|B) = \frac{P(B|R) \cdot P(R)}{P(B)}$$

To solve for the denominator, we use the law of total probability:

$$P(B) = P(R) \cdot P(B|R) + P(R^C) \cdot P(B|R^C)$$

So, we have

$$P(R|B) = \frac{P(B|R) \cdot P(R)}{P(R) \cdot P(B|R) + P(R^C) \cdot P(B|R^C)}$$

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We are given in the problem the probability of biking given raining or not raining, and the probability of it raining. So, we know the following:

$$P(B|R) = 0.2$$

$$P(B|R^C) = 0.6$$

$$P(R) = 0.7$$

We know that

$$\begin{aligned} P(R^C) &= 1 - P(R) && (1) \\ &= 1 - 0.7 && (2) \\ &= 0.3 && (3) \end{aligned}$$

0 - Bayes' Rule Practice

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Plugging in these values, we get that

$$
\begin{aligned}
P(R|B) &= \frac{P(B|R) \cdot P(R)}{P(R) \cdot P(B|R) + P(R^C) \cdot P(B|R^C)} && (4) \\
&= \frac{0.2 \cdot 0.7}{0.2 \cdot 0.7 + 0.6 \cdot 0.3} && (5) \\
&= \frac{0.14}{0.14 + 0.18} && (6) \\
&= 0.4375 && (7)
\end{aligned}
$$

Review Questions

Question #1

The range of a random variable X is the set of probabilities corresponding to the possible values X can take on.

- True
- False

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- True
- False

Question #2

Let X be the random variable representing the outcome of taking the sum of a 3-dice roll of 6-sided dice. Which function would you use to determine the probability that $X = 7$?

- CDF (cumulative distribution function)
- PMF (probability mass function)

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Question #3

Let X be the random variable representing the outcome of taking the sum of a 3-dice roll of 6-sided dice. Which function would you use to determine the probability that $X \leq 7$?

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Question #3

Let X be the random variable representing the outcome of taking the sum of a 3-dice roll of 6-sided dice. Which function would you use to determine the probability that $X \leq 7$?

- CDF (cumulative distribution function)
- PMF (probability mass function)

Question #4

A random variable X has the PMF

$$p_X(x) = \begin{cases} \frac{1}{4}, & \text{if } x = -1, \\ \frac{1}{4}, & \text{if } x = 0, \\ \frac{1}{2}, & \text{if } x = 2, \\ 0, & \text{otherwise.} \end{cases}$$

What is $E[X]$?

- -1/4
- 3/4
- 1
- 2

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What is $E[X]$?

- -1/4
- 3/4
- 1
- 2

Problem 4 – Kit Kats Again

4 – Kit Kats Again

We have N candies in the jar.
We have K kit kats in the jar.

We are drawing without replacement until we have k kit kats.

$$k \leq K \leq N$$

Let X be the number of draws until the kth kit kat (this includes the kth kit kat).

What is Ω_X, the range of X?
What is $p_X(n) = \mathbb{P}(X = n)$?

Work on finding the range of X with the people around you!

4 – Kit Kats Again

We have N candies in the jar. We have K kit kats in the jar. We are drawing without replacement until we have k kit kats. $k \leq K \leq N$

Let X be the number of draws until the kth kit kat (this includes the kth kit kat).

What is Ω_X, the range of X?

Min:
Max:
In between:

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What is Ω_X, the range of X?

Min: **k** (we were just really lucky and kept picking kit kats one after the other until we had all k that we wanted)
Max:
In between:

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Let X be the number of draws until the kth kit kat (this includes the kth kit kat).

What is Ω_X, the range of X?

Min: **k** (we were just really lucky and kept picking kit kats one after the other until we had all k that we wanted)
Max: **N – K + k** (ended up pick out all the non kit kats (N - K) in the process of picking out our k kit kats)
In between:

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Let X be the number of draws until the kth kit kat (this includes the kth kit kat).

What is Ω_X, the range of X?

Min: **k** (we were just really lucky and kept picking kit kats one after the other until we had all k that we wanted)
Max: **N – K + k** (ended up pick out all the non kit kats (N - K) in the process of picking out our k kit kats)
In between: Any integer value in between

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Let X be the number of draws until the kth kit kat (this includes the kth kit kat).

What is Ω_X, the range of X?

$$\Omega_X = \{k, k+1, \dots, N-K+k\}$$

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Let X be the number of draws until the kth kit kat (this includes the kth kit kat).

What is Ω_X, the range of X? $\Omega_X = \{k, k+1, \dots, N-K+k\}$
What is $p_X(n) = \mathbb{P}(X = n)$?

Work on finding the PMF of X with the people around you!

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Consider all N candies to be arranged randomly in a row. This is the order in which we will be drawing our n candies.

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Consider all N candies to be arranged randomly in a row. This is the order in which we will be drawing our n candies.

We know that the nth candy is a kitkat (the kth kitkat) to be chosen. This nth candy divides the row into 2 sections.

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We have N candies in the jar. We have K kit kats in the jar. We are drawing without replacement until we have k kit kats. $k \leq K \leq N$

Let X be the number of draws until the kth kit kat (this includes the kth kit kat).

What is Ω_X, the range of X? $\Omega_X = \{k, k+1, \dots, N-K+k\}$
What is $p_X(n) = \mathbb{P}(X = n)$?

Left: (?) candies and (?) kit kats
nth spot: kth kit kat
Right: (?) candies and (?) kit kats

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Let X be the number of draws until the kth kit kat (this includes the kth kit kat).

What is Ω_X, the range of X? $\Omega_X = \{k, k+1, \dots, N-K+k\}$
What is $p_X(n) = \mathbb{P}(X = n)$?

Left: **n - 1** candies and **k - 1** kit kats
nth spot: kth kit kat
Right: (?) candies and (?) kit kats

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Left: **n - 1** candies and **k - 1** kit kats
nth spot: kth kit kat
Right: **N - n** candies and **K - k** kit kats

Event:

Sample Space:

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We have N candies in the jar. We have K kit kats in the jar. We are drawing without replacement until we have k kit kats. $k \leq K \leq N$

Let X be the number of draws until the kth kit kat (this includes the kth kit kat).

What is Ω_X, the range of X? $\Omega_X = \{k, k+1, \ldots, N-K+k\}$
What is $p_X(n) = \mathbb{P}(X = n)$?

Left: **n - 1** candies and **k - 1** kit kats
nth spot: kth kit kat
Right: **N - n** candies and **K - k** kit kats

Event: number of ways to arrange the candies to have this outcome
Sample Space: number of ways to arrange the candies

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Let X be the number of draws until the kth kit kat (this includes the kth kit kat).

What is Ω_X, the range of X? $\Omega_X = \{k, k+1, \dots, N-K+k\}$
What is $p_X(n) = \mathbb{P}(X = n)$?

Left: **n - 1** candies and **k - 1** kit kats
nth spot: kth kit kat
Right: **N - n** candies and **K - k** kit kats

|E|: $\binom{n-1}{k-1}\binom{N-n}{K-k}$

|S|: $\binom{N}{K}$

4 – Kit Kats Again

We have N candies in the jar. We have K kit kats in the jar. We are drawing without replacement until we have k kit kats. $k \leq K \leq N$

Let X be the number of draws until the kth kit kat (this includes the last kit kat).

What is Ω_X, the range of X? $\Omega_X = \{k, k+1, \ldots, N-K+k\}$

What is $p_X(n) = \mathbb{P}(X = n)$? $p_X(n) = \mathbb{P}(X = n) = \frac{\binom{n-1}{k-1}\binom{N-n}{K-k}}{\binom{N}{K}}$ if $n \in \Omega_X$ and 0 otherwise

Justification: Consider all N candies to be arranged randomly in a row. We will treat the first n candies to be the "chosen" candies. We know that the nth candy is a kitkat (the kth kitkat) to be chosen. This nth candy divides the row into 2 sections. On the left, are the $n-1$ candies that were chosen (all candies chosen except the last kitkat). On the right, are the candies remaining in the jar. So the first term in the numerator is choosing the $k-1$ spots where kitkats will be chosen from the $n-1$ spots. The second term is choosing the $K-k$ spots for the kitkats that remain in the jar among the remaining $N-n$ candies. The denominator is the total number of possible ways to place the K kitkats among N candies.

Problem 5 – Hungry Washing Machine

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You have 10 pairs of socks (so 20 socks in total), with each pair being a different color. You put them in the washing machine, but the washing machine eats 4 of the socks chosen at random. Every subset of 4 socks is equally probable to be the subset that gets eaten. Let X be the number of complete pairs of socks that you have left.

a) What is the range of X, Ω_X (the set of possible values it can take on)? What is the probability mass function of X?

b) Find $\mathbb{E}[X]$ from the definition of expectation.

Work on this problem with the people around you, and then we'll go over it together!

5 – Hungry Washing Machine

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The washing machine eats 4 socks every time. It can either eat a single sock from 4 pairs of socks, leaving us with 6 complete pairs, or a single sock from 2 pairs and a matching pair, leaving us with 7 complete pairs, or 2 pairs of matching socks, leaving us with 8 complete pairs.

$$\Omega_X = \{6,7,8\}$$

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$$\Omega_X = \{6,7,8\}$$

We are dealing with a sample space with equally likely outcomes. As such, we can compute use the formula $\mathbb{P}(E) = \frac{|E|}{|\Omega|}$. We know that $|\Omega| = \binom{20}{4}$ because the washing machine picks a set of 4 socks out of 20 possible socks.

5 – Hungry Washing Machine

a) What is the range of X, Ω_X (the set of possible values it can take on)? What is the probability mass function of X?

To define the pmf of X, we consider each value in the range of X.

For $k = 6$, we first pick 4 out of 10 pairs of socks from which we will eat a single sock ($\binom{10}{4}$ ways), and for each of these 4 pairs we have two socks to pick from ($\binom{2}{1}^4$ ways). Using the product rule, we get $|X = 6| = \binom{10}{4}2^4$

For $k = 7$, we first pick 1 out of 10 pairs of socks to eat in its entirety ($\binom{10}{1}$ ways), and then 2 out of the 9 remaining pairs from which we will eat a single sock ($\binom{9}{2}$ ways), and for each of these 2 pairs we have two socks to pick from ($\binom{2}{1}^2$ ways). Using the product rule, we get $|X = 7| = 10\binom{9}{2}2^2$

For $k = 8$, we pick 2 out of 10 pairs of socks to eat ($\binom{10}{2}$ ways). We get $|X = 8| = \binom{10}{2}$

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For $k = 8$, we pick 2 out of 10 pairs of socks to eat ($\binom{10}{2}$ ways). We get $|X = 8| = \binom{10}{2}$

$$p_X(k) = \begin{cases} \frac{\binom{10}{4}2^4}{\binom{20}{4}} & k = 6 \\ \frac{10\binom{9}{2}2^2}{\binom{20}{4}} & k = 7 \\ \frac{\binom{10}{2}}{\binom{20}{4}} & k = 8 \\ 0 & otherwise \end{cases}$$

5 – Hungry Washing Machine

b) Find $\mathbb{E}[X]$ from the definition of expectation.

5 – Hungry Washing Machine

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$$\mathbb{E}[X] = \sum_{k \in \Omega_X} k \cdot p_X(k) = 6 \cdot \frac{\binom{10}{4} 2^4}{\binom{20}{4}} + 7 \cdot \frac{10 \binom{9}{2} 2^2}{\binom{20}{4}} + 8 \cdot \frac{\binom{10}{2}}{\binom{20}{4}} = \frac{120}{19}$$

Problem 2 – Identify that Range!

2 – Identify that Range!

Identify the support/range Ω_X of the random variable X, if X is...

a) The sum of two rolls of a six-sided die.

b) The number of lottery tickets I buy until I win it.

c) The number of heads in n flips of a coin with $0 < \mathbb{P}(\text{head}) < 1$.

d) The number of heads in n flips of a coin with $\mathbb{P}(\text{head}) = 1$.

e) The number of whole minutes I wait at the bus stop for the next bus.

Work on this problem with the people around you, and then we'll go over it together!

2 – Identify that Range!

Identify the support/range Ω_X of the random variable X, if X is...

a) The sum of two rolls of a six-sided die.

Min:

Max:

In between?

2 – Identify that Range!

Identify the support/range Ω_X of the random variable X, if X is...

a) The sum of two rolls of a six-sided die.

X takes on every integer value between the min sum 2, and the max sum 12.

$$\Omega_X = \{2, 3, \dots, 12\}$$

2 – Identify that Range!

Identify the support/range Ω_X of the random variable X, if X is...

b) The number of lottery tickets I buy until I win it.

Min:

Max:

In between?

2 – Identify that Range!

Identify the support/range Ω_X of the random variable X, if X is...

b) The number of lottery tickets I buy until I win it.

X takes on all positive integer values (I may never win the lottery).

$$\Omega_X = \{1, 2, \ldots\}$$

2 – Identify that Range!

Identify the support/range Ω_X of the random variable X, if X is...

c) The number of heads in n flips of a coin with $0 < \mathbb{P}(\text{head}) < 1$.

Min:

Max:

In between?

2 – Identify that Range!

Identify the support/range Ω_X of the random variable X, if X is...

c) The number of heads in n flips of a coin with $0 < \mathbb{P}(\text{head}) < 1$.

X takes on every integer value between the min number of heads 0, and the max n.

$$\Omega_X = \{1, 2, \dots, n\}$$

2 – Identify that Range!

Identify the support/range Ω_X of the random variable X, if X is...

d) The number of heads in n flips of a coin with $\mathbb{P}(\text{head}) = 1$.

Min:

Max:

In between?

2 – Identify that Range!

Identify the support/range Ω_X of the random variable X, if X is...

d) The number of heads in n flips of a coin with $\mathbb{P}(\text{head}) = 1$.

Since $\mathbb{P}(\text{head}) = 1$, we are guaranteed to get n heads in n flips..

$$\Omega_X = \{n\}$$

2 – Identify that Range!

Identify the support/range Ω_X of the random variable X, if X is...

e) The number of whole minutes I wait at the bus stop for the next bus.

Min:

Max:

In between?

2 – Identify that Range!

Identify the support/range Ω_X of the random variable X, if X is...

e) The number of whole minutes I wait at the bus stop for the next bus.

The number of whole minutes is discrete and will take on values between the minimum waiting time (0, the bus is here), and the maximum waiting time (∞, the bus never gets here).

$$\Omega_X = \{0, 1, \dots\}$$

Review of Main Concepts

- **Random Variable (rv):** A numeric function $X : \Omega \to \mathbb{R}$ of the outcome.
- **Range/Support:** The support/range of a random variable X, denoted Ω_X, is the set of all possible values that X can take on.
- **Discrete Random Variable (drv):** A random variable taking on a countable (either finite or countably infinite) number of possible values.
- **Probability Mass Function (pmf)** for a discrete random variable X: a function $p_X : \Omega_X \to [0,1]$ with $p_X(x) = \mathbb{P}(X = x)$ that maps possible values of a discrete random variable to the probability of that value happening, such that $\sum_x p_X(x) = 1$.
- **Cumulative Distribution Function (CDF)** for a random variable X: a function $F_X : \mathbb{R} \to \mathbb{R}$ with $F_X(x) = \mathbb{P}(X \le x)$

Review of Main Concepts

- **Expectation (expected value, mean, or average):** The expectation of a discrete random variable is defined to be
$$\mathbb{E}[X] = \sum_x x\, p_X(x) = \sum_x x\, \mathbb{P}(X = x)$$
The expectation of a function of a discrete random variable $g(X)$ is
$$\mathbb{E}[g(X)] = \sum_x g(x)\, p_X(x)$$

Review of Main Concepts

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$$\mathbb{E}[X] = \sum_x x\, p_X(x) = \sum_x x\, \mathbb{P}(X = x)$$

The expectation of a function of a discrete random variable $g(X)$ is

$$\mathbb{E}[g(X)] = \sum_x g(x)\, p_X(x)$$

- **Linearity of Expectation:** Let X and Y be random variables, and $a, b, c \in \mathbb{R}$. Then, $\mathbb{E}[aX + bY + c] = a\mathbb{E}[X] + b\mathbb{E}[Y] + c$. Also, for any random variables $X_1, \dots, X_n$,

$$\mathbb{E}[X_1 + X_2 + \cdots + X_n] = \mathbb{E}[X_1] + \mathbb{E}[X_2] + \cdots + \mathbb{E}[X_n]$$

Review of Main Concepts

- **Variance:** Let X be a random variable and $\mu = E[X]$. The variance of X is defined to be

  $$Var(X) = \mathbb{E}[(X - \mu)^2]$$

  Notice that since this is an expectation of a non-negative random variable ($(X - \mu)^2$), variance is always nonnegative. With some algebra, we can simplify this to

  $$Var(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

- **Standard Deviation:** Let X be a random variable. We define the standard deviation of X to be the square root of the variance, and denote it $\sigma = \sqrt{Var(X)}$

- **Property of Variance:** Let $a, b \in \mathbb{R}$ and let X be a random variable. Then,

  $$Var(aX + b) = a^2 Var(X)$$

Review of Main Concepts

- **Independence:** Random variables X and Y are independent iff

$$\forall x \forall y, \mathbb{P}(X = x \cap Y = y) = \mathbb{P}(X = x)\,\mathbb{P}(Y = y)$$

  In this case, we have $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$ (the converse is not necessarily true).

- **i.i.d. (independent and identically distributed):** Random variables $X_1, \dots, X_n$ are i.i.d. (or iid) iff they are independent and have the same probability mass function..

- **Variance of Independent Variables:** If X is independent of Y,

$$Var(X + Y) = Var(X) + Var(Y)$$

  This depends on independence, whereas linearity of expectation always holds.
  Note that this combined with the above shows that $\forall a, b, c \in \mathbb{R}$ and if X is independent of Y,

$$Var(aX + bY + c) = a^2 Var(X) + b^2 Var(Y)$$

That’s All, Folks!

Thanks for coming to section this week!
Any questions?

Problem 7 – Frogger

7 – Frogger

A frog starts on a 1-dimensional number line at 0. At each second, independently, the frog takes a unit step right with probability p_1, to the left with probability p_2, and doesn't move with probability p_3, where $p_1 + p_2 + p_3 = 1$. After 2 seconds, let X be the location of the frog.

a) Find $p_X(k)$, the probability mass function for X.

b) Compute $\mathbb{E}[X]$ from the definition.

Work on this problem with the people around you, and then we'll go over it together!

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a) Find $p_X(k)$, the probability mass function for X.

Let L be a left step, R be a right step, and N be no step.

The range of X is $\{-2, -1, 0, 1, 2\}$. We can compute:

$p_X(-2) = \mathbb{P}(X = -2) = \mathbb{P}(LL) = p_2^2$

$p_X(-1) = \mathbb{P}(X = -1) = \mathbb{P}(LN \cup NL) = 2p_2p_3$

$p_X(0) = \mathbb{P}(X = 0) = \mathbb{P}(NN \cup LR \cup RL) = p_3^2 + 2p_1p_2$

$p_X(1) = \mathbb{P}(X = 1) = \mathbb{P}(RN \cup NR) = 2p_1p_3$

$p_X(2) = \mathbb{P}(X = 2) = \mathbb{P}(RR) = p_1^2$

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$$p_X = \begin{cases} p_2^2 & k = -2 \\ 2p_2p_3 & k = -1 \\ p_3^2 + 2p_1p_2 & k = 0 \\ 2p_1p_3 & k = 1 \\ p_1^2 & k = 2 \\ 0 & \text{otherwise} \end{cases}$$

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b) Compute $\mathbb{E}[X]$ from the definition.

$$\mathbb{E}[X] = (-2)(p_2^2) + (-1)(2p_2p_3) + (0)(p_3^2 + 2p_1p_2) + (1)(2p_1p_3) + (2)(p_1^2) = 2(p_1 - p_2)$$