Markov Chains

Example 1: Suppose I flip a biased coin which is heads w.p. p

Let X = # of flips until I see my first heads. $\boxed{E[X] = ?}$

Zoo/Reference Sheet:

Alternative calculation:

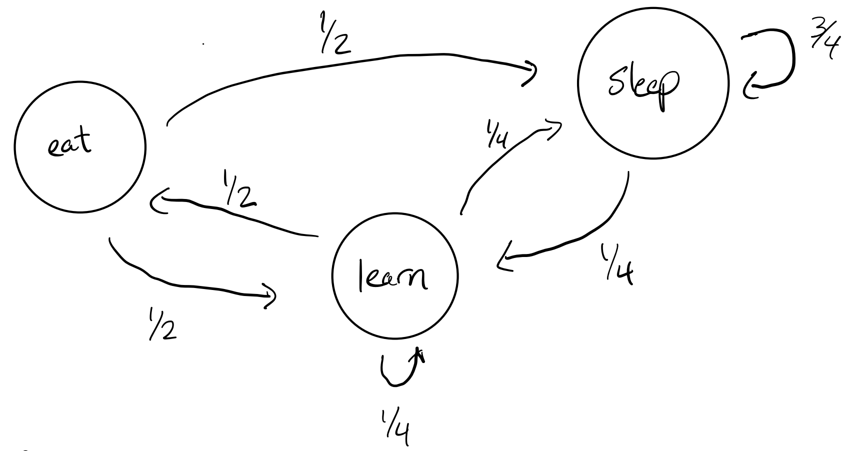
Example 2: Same setup, but let Y = # of flips until I see 2 heads in a row

$\boxed{E[Y] = ?}$

idea 1: geometric RV?

idea 2: condition on most recent flip?

Example 3



At each time step t:

I am in one of four states:

$\{$ gym, eat, sleep, learn $\}$

$\Pr[$ in state s' at time $t+1$ $|$ in state s at time t $]$

$=$

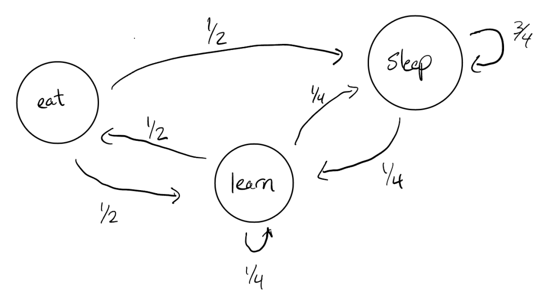
This random process is called a **Markov Chain**

remark: Looks kind of like DFA/NFA from 311 .... except labels are probabilities, not input symbols.

Define random variable X_t = state we are in at time t.

Questions 1) $\Pr[X_1 = \text{sleep} \mid X_0 = \text{sleep}] =$

2) $\Pr[X_2 = \text{sleep} \mid X_0 = \text{sleep}] =$



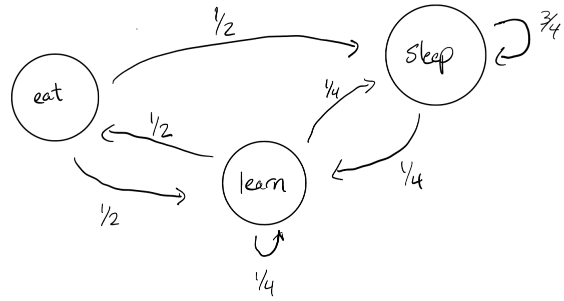
3) $\Pr[X_t = \text{sleep} \mid X_0 = \text{sleep}] =$?

4) How long do I expect to wait before I get to eat? (assuming still $X_0 = \text{Sleep}$)

Y_S = amount of time until eat starting in Sleep

Y_L = amount of time until eat starting from Learn

Y_E = amount of time until eat starting from Eat



LTE: $E[Y_S] =$

"first-step equations"

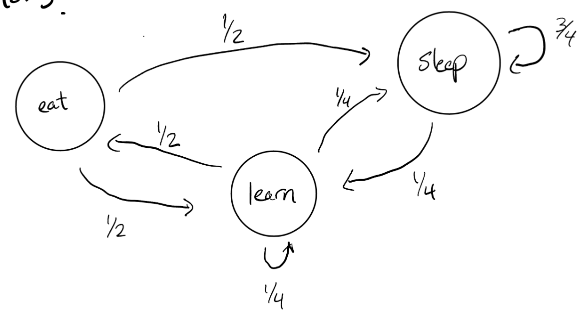
Definition A Markov Chain is a sequence of random variables $X_0, X_1, X_2, \ldots$ satisfying

1) $\Pr[X_{t+1} = s_{t+1} \mid X_t = s_t, \ldots X = s_0] = \Pr[X_{t+1} = s_{t+1} \mid X_t = s_t]$

2) $\Omega_{X_i} = S$ (for example, $S = \{$eat, sleep, learn$\}$)

3) p_{X_0} is an arbitrary pmf on S

Let's write PMF's for X_t's as (row) vectors.



$$\begin{array}{cccc} & \text{eat} & \text{sleep} & \text{learn} \\ q^{(0)} = & [\,0, & 1, & 0\,] \end{array}$$

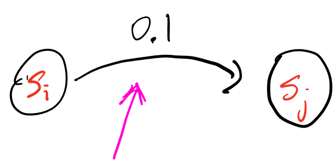
$$q^{(1)} = [\qquad\qquad\qquad]$$

$$q^{(2)} = [\qquad\qquad\qquad]$$

Question: How to get $q^{(t)}$ (PMF of X_t) from $\underbrace{q^{(0)}, q^{(1)}, \ldots, q^{(t-1)}}_{\text{PMF's for } X_0, X_1, \ldots, X_{t-1}}$?

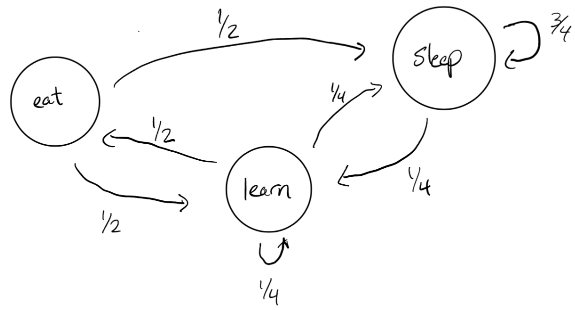
Definition Let $X_0, X_1, \ldots$ be a markov chain over state space $S = \{s_1, \ldots, s_n\}$. Define the **transition probability matrix** P to be the unique matrix s.t.

$$P_{ij} = \Pr[X_t = s_j \mid X_{t-1} = s_i]$$



$P_{ij} = 0.1$ in this example

example :



$$P = \begin{array}{c} \\ \text{eat} \\ \text{sleep} \\ \text{learn} \end{array} \begin{array}{c} \begin{array}{ccc} \text{eat} & \text{sleep} & \text{learn} \end{array} \\ \begin{bmatrix} 0 & 1/2 & 1/2 \\ 0 & 3/4 & 1/4 \\ 1/2 & 1/4 & 1/4 \end{bmatrix} \end{array}$$

$$q^{(t)}_{eat} = \Pr[X_t = eat] = \Pr[X_t = eat \cap X_{t-1} = eat]$$

$$\Pr[X_t = eat \cap X_{t-1} = sleep] + \Pr[X_t = eat \cap X_{t-1} = learn]$$

observation $\quad q^{(t)} = q^{(t-1)} P$

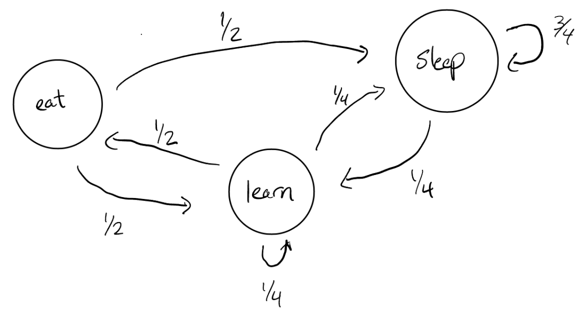
claim $\quad q^{(t)} = q^{(0)} P^{t}$

proof

question What does $q^{(t)}$ look like as $t \to \infty$?

equivalently: what does $q^{(0)} P^{t}$ look like as $t \to \infty$?

$$
\begin{array}{cccc}
 & \text{eat} & \text{sleep} & \text{learn} \\
q^{(0)} = & [\,0, & 1, & 0\,] \\
q^{(1)} = & [\,0, & 3/4, & 1/4\,] \\
q^{(2)} = & [\,1/8, & 5/8, & 1/4\,] \\
q^{(3)} = & [\,1/8, & 19/32, & 9/32\,] \\
q^{(4)} = & [\,9/64, & 37/64, & 9/32\,] \\
\vdots & \vdots & \vdots & \vdots
\end{array}
$$



$$
P = \begin{array}{c|ccc}
 & \text{eat} & \text{sleep} & \text{learn} \\
\hline
\text{eat} & 0 & 1/2 & 1/2 \\
\text{sleep} & 0 & 3/4 & 1/4 \\
\text{learn} & 1/2 & 1/4 & 1/4
\end{array}
$$

$$q^{(10)} \approx [0.142853,\ 0.571442,\ 0.285706]$$

$$q^{(11)} = [0.142853,\ 0.571434,\ 0.285713]$$

observation if $q^{(t+1)} = q^{(t)}$ then distribution never changes again!

Called a **stationary distribution**

solution to $\pi = \pi P$ so a (left) eigenvector with eigenvalue $\underline{1}$!

Let's solve!

$$\pi = \pi P$$

$$[\pi_E, \pi_S, \pi_L] = [\pi_E, \pi_S, \pi_L] \begin{bmatrix} 0 & 1/2 & 1/2 \\ 0 & 3/4 & 1/4 \\ 1/2 & 1/4 & 1/4 \end{bmatrix}$$

Big Theorem for Markov Chains For any sufficiently nice[(*)] Markov Chain, no matter what $q^{(0)}$ is,

$$\lim_{t \to \infty} q^{(0)} P^t = \lim_{t \to \infty} q^{(t)} = \pi$$

solution to $\pi = \pi P$

(*): "nice" being irreducible, aperiodic, and positive recurrent.