Variance

CSE 312 Autumn 25
Lecture 12

Variance

Where are we?

A random variable is a way to summarize what outcome you saw.

The Expectation of a random variable is its average value.

A way to summarize a random variable

Variance

Another one number summary of a random variable.

But wait, we already have expectation, what’s this for?

Consider these two games

Would you be willing to play these games?

Game 1: I will flip a fair coin; if it’s heads, I pay you \$1. If it’s tails, you pay me \$1. Let X_1 be your profit if you play game 1

Game 2: I will flip a fair coin; if it’s heads, I pay you \$10,000. If it’s tails, you pay me \$10,000. Let X_2 be your profit if you play game 2.

Both games are “fair” ($\mathbb{E}[X_1] = \mathbb{E}[X_2] = 0$)

What’s the difference

Expectation tells you what the average will be…

But it doesn’t tell you how “extreme” your results could be.

Nor how likely those extreme results are.

Game 2 has many (well, only) very extreme results.

In expectation they “cancel out” but if you can only play once…

…it would be nice to measure that.

Designing a Measure – Try 1

Well let's measure how far all the events are away from the center, and how likely they are

$\sum_{\omega} \mathbb{P}(\omega) \cdot (X(\omega) - \mathbb{E}[X])$

What happens with Game 1?

$$\frac{1}{2} \cdot (1 - 0) + \frac{1}{2} \cdot (-1 - 0)$$
$$\frac{1}{2} - \frac{1}{2} = 0$$

What happens with Game 2?

$$\frac{1}{2} \cdot (100000 - 0) + \frac{1}{2} \cdot (-100000 - 0)$$
$$5000 - 5000 = 0$$

Designing a Measure – Try 2

How do we prevent cancelling? Squaring makes everything positive.

$$\sum_\omega \mathbb{P}(\omega) \cdot (X(\omega) - \mathbb{E}[X])^2$$

What happens with Game 1?

$$\frac{1}{2} \cdot (1 - 0)^2 + \frac{1}{2} \cdot (-1 - 0)^2$$

$$\frac{1}{2} + \frac{1}{2} = 1$$

What happens with Game 2?

$$\frac{1}{2} \cdot (100000 - 0)^2 + \frac{1}{2} \cdot (-100000 - 0)^2$$

$$5{,}000{,}000{,}000 + 5{,}000{,}000{,}000 = 10^{10}$$

Why Squaring

Why not absolute value? Or Fourth power?

Squaring is nicer algebraically.

Our goal with variance was to talk about the spread of results. Squaring makes extreme results even more extreme.

Fourth power over-emphasizes the extreme results (for our purposes).

Variance

Variance

The variance of a random variable X is

$$\mathrm{Var}(X) = \sum_{\omega} \mathbb{P}(\omega) \cdot (X(\omega) - \mathbb{E}[X])^2 \; = \mathbb{E}[(X - \mathbb{E}[X])^2] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

The first two forms are the definition. The last one is an algebra trick.

Variance of a die

Let X be the result of rolling a fair die.

$\mathrm{Var(X)} = \mathbb{E}[(X - \mathbb{E}[X])^2] = \mathbb{E}[(X - 3.5)^2]$

$= \frac{1}{6}(1 - 3.5)^2 + \frac{1}{6}(2 - 3.5)^2 + \frac{1}{6}(3 - 3.5)^2 + \frac{1}{6}(4 - 3.5)^2 + \frac{1}{6}(5 - 3.5)^2 + \frac{1}{6}(6 - 3.5)^2$

$= \frac{35}{12} \approx 2.92.$

Or $\mathbb{E}[X^2] - (E[X])^2 = \sum_{k=1}^{6} \frac{1}{6} \cdot k^2 - 3.5^2 = \frac{91}{6} - 3.5^2 \approx 2.92$

Variance of n Coin Flips

Flip a coin n times, where it comes up heads with probability p each time (independently). Let X be the total number of heads.

We'll see next time $\mathbb{E}[X] = np$.

Also define: $X_i = \begin{cases} 1 & \text{if flip } i \text{ is heads} \\ 0 & \text{otherwise} \end{cases}$

Variance of n Coin Flips

Flip a coin n times, where it comes up heads with probability p each time (independently). Let X be the total number of heads.

What about $\text{Var}(X)$

$\mathbb{E}[(X - \mathbb{E}[X])^2] = \sum_{\omega} \mathbb{P}(\omega)(X(\omega) - np)^2$

$= \sum_{k=0}^{n} \binom{n}{k} \cdot p^k (1-p)^{n-k} \cdot (k - np)^2$

Algebra time?

Variance

If X and Y are independent then
$$\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$$

Are the X_i independent? Yes!

In this problem X_i is independent of X_j for $i \neq j$ where

$$X_i = \begin{cases} 1 & \text{if flip } i \text{ was heads} \\ 0 & \text{otherwise} \end{cases}$$

Variance

$\mathrm{Var}(X) = \mathrm{Var}(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} \mathrm{Var}(X_i)$

What's the $\mathrm{Var}(X_i)$?

$\mathbb{E}[(X_i - \mathbb{E}[X_i])^2]$

$= \mathbb{E}[(X_i - p)^2]$

$= p(1-p)^2 + (1-p)(0-p)^2$

$= p(1-p)[(1-p) + p] = p(1-p).$

OR $Var(X_i) = \mathbb{E}\left[X_i^2\right] - \mathbb{E}[X_i]^2 = \mathbb{E}[X_i] - p^2 = p - p^2 = p(1-p)$.

Plugging In

$\mathrm{Var}(X) = \mathrm{Var}(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} \mathrm{Var}(X_i)$

What's the $\mathrm{Var}(X_i)$?

$p(1-p)$.

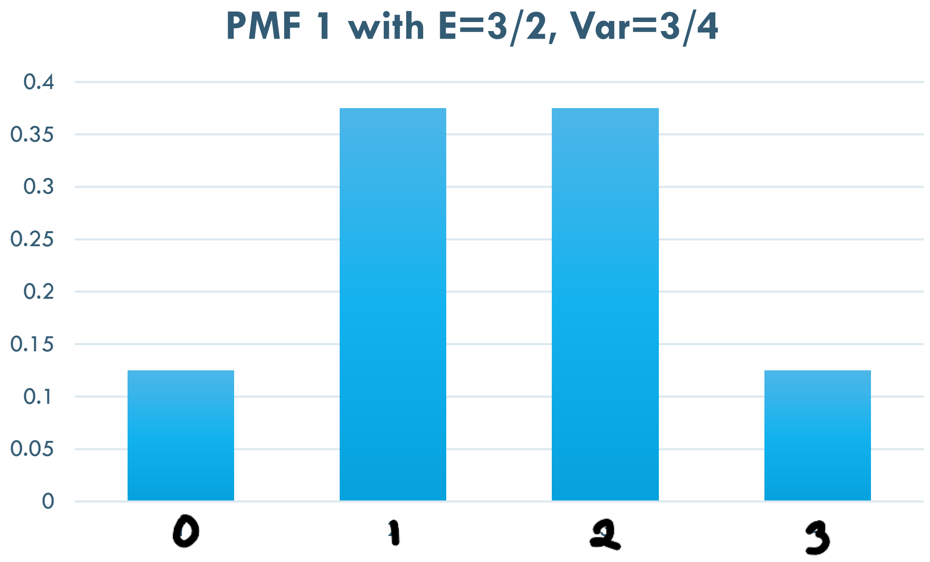
$\mathrm{Var}(X) = \sum_{i=1}^{n} p(1-p) = np(1-p)$.

Expectation and Variance aren't everything

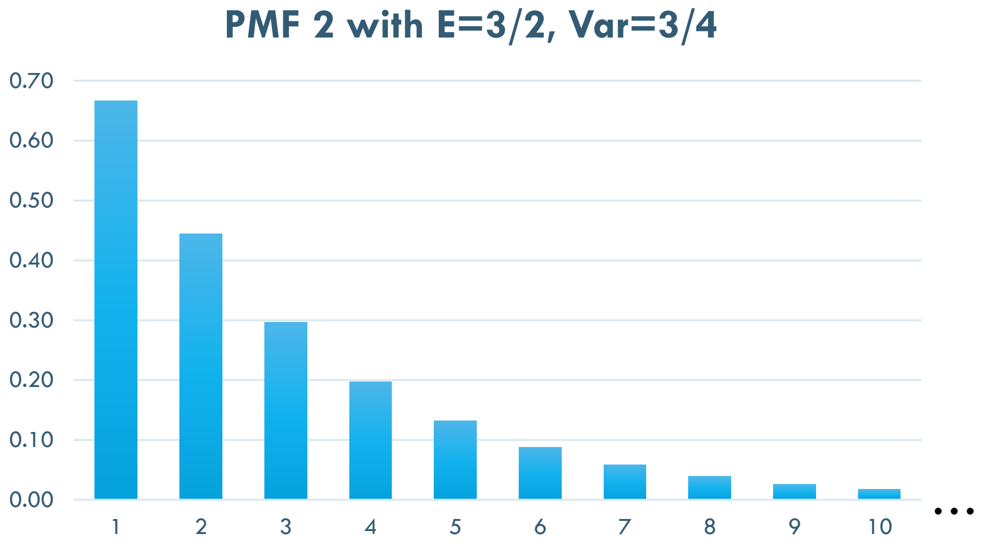
Alright, so expectation and variance is everything right?

No!

Flip a fair coin 3 times indep. Count heads.



Flip a biased coin (prob heads=2/3) until heads. Count flips.



A PMF or CDF *does* fully describe a random variable.

Proof of Calculation Trick

$\mathbb{E}[(X - \mathbb{E}[X])^2] = \mathbb{E}[X^2 - 2X\mathbb{E}[X] + (\mathbb{E}[X])^2]$ expanding the square

$= \mathbb{E}[X^2] - \mathbb{E}\big[2X\mathbb{E}[X]\big] + \mathbb{E}[(\mathbb{E}[X])^2]$ linearity of expectation.

$= \mathbb{E}[X^2] - 2\mathbb{E}[X]\mathbb{E}[X] + \mathbb{E}[(\mathbb{E}[X])^2]$ linearity of expectation.

$= \mathbb{E}[X^2] - 2\mathbb{E}[X]\mathbb{E}[X] + (\mathbb{E}[X])^2$ expectation of a constant is the constant

$= \mathbb{E}[X^2] - 2(\mathbb{E}[X])^2 + (\mathbb{E}[X])^2$

$= \mathbb{E}[X^2] - (\mathbb{E}[X])^2$

So $\mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$.