

# Linearity of Expectation

CSE 312 Autumn 25  
Lecture 11

# Outline

Functions of Random Variables

□ Linearity of expectation

Statement

Proof

A whole bunch of examples

# Functions of a random variable

Let  $X, Y$  be random variables defined on the same sample space.

Functions of  $X$  and/or  $Y$  like

$$\cancel{X + Y}$$

$$X^2$$

$$\underline{2X + 3}$$

Etc.

**Are** random variables! (Say what the outcome is, and these functions give you a number. They're functions from  $\Omega \rightarrow \mathbb{R}$ . **That's the definition of a random variable!**)

# Expectations of functions of random variables

Let's say we have a random variable  $X$  and a function  $g$ . What is  $\mathbb{E}[g(X)]$ ?

$$\mathbb{E}[g(X)] = \sum_{\omega \in \Omega} g(X(\omega)) \cdot \mathbb{P}(\omega)$$

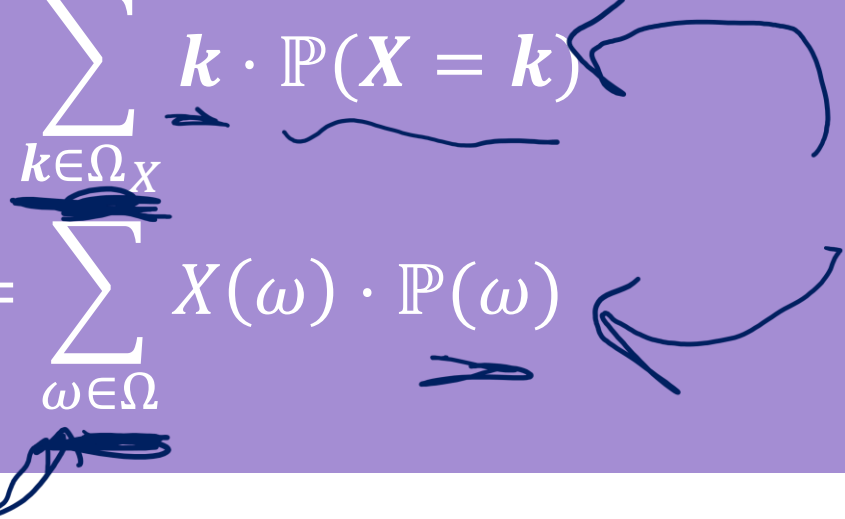
Equivalently:  $\mathbb{E}[g(X)] = \sum_{k \in \Omega_{g(X)}} k \cdot \mathbb{P}(g(X) = k)$

Notice that  $\mathbb{E}[g(X)]$  might not be  $g(\mathbb{E}[X])$ .

# Expectation

## Expectation

The “expectation” (or “expected value”) of a random variable  $X$  is:

$$\mathbb{E}[X] = \sum_{k \in \Omega_X} k \cdot \mathbb{P}(X = k)$$

$$\mathbb{E}[X] = \sum_{\omega \in \Omega} X(\omega) \cdot \mathbb{P}(\omega)$$

Intuition: The weighted average of values  $X$  could take on.  
Weighted by the probability you actually see them.

# Linearity of Expectation

## Linearity of Expectation

For any two random variables  $X$  and  $Y$ :

$$\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$$

Note:  $X$  and  $Y$  do not have to be independent

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Extending this to  $n$  random variables,  $X_1, X_2, \dots, X_n$

$$\mathbb{E}[X_1 + X_2 + \dots + X_n] = \mathbb{E}[X_1] + \mathbb{E}[X_2] + \dots + \mathbb{E}[X_n]$$


This can be proven by induction.

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Constants are also fine:

For real numbers  $a, b, c$

$$\begin{aligned} \mathbb{E}[aX + bY + c] &= \mathbb{E}[aX] + \mathbb{E}[bY + c] \\ &= \underbrace{a\mathbb{E}[X] + b\mathbb{E}[Y] + c} \end{aligned}$$

# Linearity of Expectation - Proof

## Linearity of Expectation

For any two random variables  $X$  and  $Y$ :

$$\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$$

Note:  $X$  and  $Y$  do not have to be independent

Proof:

$$\begin{aligned}\mathbb{E}[X + Y] &= \sum_{\omega \in \Omega} \mathbb{P}(\omega) (X(\omega) + Y(\omega)) \\ &= \sum_{\omega \in \Omega} \mathbb{P}(\omega) X(\omega) + \sum_{\omega \in \Omega} \mathbb{P}(\omega) Y(\omega) \\ &= \sum_{\omega \in \Omega} \mathbb{P}(\omega) X(\omega) + \sum_{\omega \in \Omega} \mathbb{P}(\omega) Y(\omega) \\ &= \mathbb{E}[X] + \mathbb{E}[Y]\end{aligned}$$

# Fishy Business

Say you and your friend go fishing everyday.

- You catch  $X$  fish, with  $\mathbb{E}[X] = 3$
- Your friend catches  $Y$  fish, with  $\mathbb{E}[Y] = 7$
- How many fish do both of you bring on an average day?

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Let  $Z$  be the r.v. representing the total number of fish you both catch

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What is your average profit?

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$$\mathbb{E}[10Z - 15] = 10\mathbb{E}[Z] - 15 = 100 - 15 = 85$$

# Coin Tosses

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$$p_Y(y) = \begin{cases} \frac{1}{4} & \text{if } y = 0 \\ \frac{1}{2} & \text{if } y = 1 \\ \frac{1}{4} & \text{if } y = 2 \\ 0 & \text{otherwise} \end{cases}$$

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$$\mathbb{E}[Y] = \sum_{k \in \Omega_Y} p_Y(k) \cdot k = \frac{1}{4} \cdot 0 + \frac{1}{2} \cdot 1 + \frac{1}{4} \cdot 2 = 1$$


# Repeated Coin Tosses

$p^n$

Now what if the probability of flipping a head was  $p$  and that we wanted to find the total number of heads flipped when we flip the coin  $n$  times?

Let  $X$  be the r.v. representing the total number of heads.

Make a prediction --- what should  $\mathbb{E}[X]$  be?

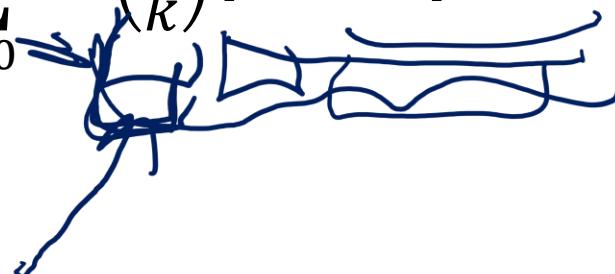
# Repeated Coin Tosses

HTHT  
 $p(1-p)p(1-p)(1-p)$

1      3

Now what if the probability of flipping a head was  $p$  and that we wanted to find the total number of heads flipped when we flip the coin  $n$  times?

Let  $X$  be the r.v. representing the total number of heads.

$$\mathbb{E}[X] = \sum_{k=0}^n k \cdot \mathbb{P}(Y = k) = \sum_{k=0}^n k \cdot \binom{n}{k} p^k (1-p)^{n-k}$$


Ok, but what actually is it?  
I don't have intuition for this  
formula.

# Repeated Coin Tosses

Now what if the probability of flipping a head was  $p$  and that we wanted to find the total number of heads flipped when we flip the coin  $n$  times?

$$\begin{aligned}\mathbb{E}[\mathbf{X}] &= \sum_{k=0}^n k \cdot \mathbb{P}(Y = k) = \sum_{k=0}^n k \cdot \binom{n}{k} p^k (1-p)^{n-k} \\ &= \sum_{k=1}^n k \cdot \binom{n}{k} p^k (1-p)^{n-k} \\ &= \sum_{k=1}^n n \cdot \binom{n-1}{k-1} p^k (1-p)^{n-k} \\ &= np \sum_{i=0}^{n-1} \binom{n-1}{i} p^i (1-p)^{n-1-i} \\ &= np(p + (1-p))^{n-1} = np\end{aligned}$$

$k \binom{n}{k} = n \binom{n-1}{k-1}$

Binomial Theorem!

We did it! And all it took was a clever application of the binomial theorem, setup by a very non-obvious application of an obscure combinatorial identity. Ezpz.

# Repeated Coin Tosses

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No one wants to do proofs like this every time!

# Linearity of Expectation

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For any two random variables  $X$  and  $Y$ :

$$\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$$

Note:  $X$  and  $Y$  do not have to be independent

Extending this to  $n$  random variables,  $X_1, X_2, \dots, X_n$

$$\mathbb{E}[X_1 + X_2 + \dots + X_n] = \mathbb{E}[X_1] + \mathbb{E}[X_2] + \dots + \mathbb{E}[X_n]$$

# Indicator Random Variables

For any event  $A$ , we can define the indicator random variable  $\mathbf{1}[A]$  for  $A$

$$\mathbf{1}[A] = \begin{cases} 1 & \text{if event } A \text{ occurs} \\ 0 & \text{otherwise} \end{cases}$$

You'll also see notation like:

$$\mathbf{1}[A], \mathbf{1}_A, \mathbf{1}[\text{some boolean}]$$

$$\begin{aligned} \mathbb{P}(X = 1) &= \mathbb{P}(A) \\ \mathbb{P}(X = 0) &= 1 - \mathbb{P}(A) \end{aligned}$$

$$p_X(x) = \begin{cases} \mathbb{P}(A) & \text{if } x = 1 \\ 1 - \mathbb{P}(A) & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} \mathbb{E}[X] &= 1 \cdot p_X(1) + 0 \cdot p_X(0) \\ &= p_X(1) = \mathbb{P}(A) \end{aligned}$$

# Repeated Coin Tosses (Again)

The probability of flipping a head is  $p$  and we want to find the total number of heads flipped when we flip the coin  $n$  times?

Let  $X$  be the total number of heads

What indicators can we define? What 'Booleans' have enough information to combine (add) and solve the problem?

# Repeated Coin Tosses (Again)

The probability of flipping a head is  $p$  and we want to find the total number of heads flipped when we flip the coin  $n$  times?

Let  $X$  be the total number of heads

Define  $X_i$  as follows:

$$X_i = \begin{cases} 1 & \text{if the } i\text{th coin flip is heads} \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned}\mathbb{P}(X_i = 1) &= p \\ \mathbb{P}(X_i = 0) &= 1 - p\end{aligned}$$

$$X = \sum_{i=1}^n X_i$$

$$\mathbb{E}[X_i] = 1 \cdot p + 0 \cdot (1 - p) = p$$

# Repeated Coin Tosses (Again)

The probability of flipping a head is  $p$  and we want to find the total number of heads flipped when we flip the coin  $n$  times?

Let  $X$  be the total number of heads

$$X = \sum_{i=1}^n X_i$$
$$\mathbb{E}[X_i] = p$$

$$\begin{aligned}\mathbb{E}[X] &= \mathbb{E}[\sum_{i=1}^n X_i] \\ &= \mathbb{E}[X_1 + X_2 + \cdots + X_n] \\ &= \mathbb{E}[X_1] + \mathbb{E}[X_2] + \cdots + \mathbb{E}[X_n] \\ &= \sum_{i=1}^n \mathbb{E}[X_i] \\ &= \sum_{i=1}^n p = np\end{aligned}$$

# Computing complicated expectations

We often use these three steps to solve complicated expectations

1. Decompose: Finding the right way to decompose the random variable into sum of simple random variables

$$\underline{X} = \underbrace{X_1 + X_2 + \cdots + X_n}$$

2. LOE: Apply Linearity of Expectation

$$\mathbb{E}[X] = \mathbb{E}[X_1] + \mathbb{E}[X_2] + \cdots + \mathbb{E}[X_n]$$

3. Conquer: Compute the expectation of each  $X_i$

Often  $X_i$  are indicator random variables

# Pairs with the same birthday $\times$ # of pairs

In a class of  $m$  students, on average how many pairs of people have the same birthday?

$$X_1, X_2, X_3, \dots, X_{\binom{m}{2}}$$

Decompose:

LOE:

Conquer:

# Pairs with the same birthday

In a class of  $m$  students, on average how many pairs of people have the same birthday?

Decompose: Let  $X$  be the number of pairs with the same birthday

Define  $X_{ij}$  as follows:

$$\underline{X_{ij}} = \begin{cases} 1 & \text{if person } i, j \text{ have the same birthday} \\ 0 & \text{otherwise} \end{cases}$$

$$\underline{X} = \sum_{i,j} X_{ij}$$

LOE:

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LOE:

$$\mathbb{E}[X] = \mathbb{E}\left[\sum_{i,j} X_{ij}\right] = \sum_{i,j} \mathbb{E}[X_{ij}]$$

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LOE:

$$\mathbb{E}[X] = \mathbb{E}[\sum_{i,j} X_{ij}] = \sum_{i,j} \mathbb{E}[X_{ij}]$$

Conquer:

$$\begin{aligned} \mathbb{E}[X_{ij}] &= \mathbb{P}(X_{ij} = 1) = \frac{365}{365 \cdot 365} = \frac{1}{365} \\ \mathbb{E}[X] &= \binom{m}{2} \cdot \mathbb{E}[X_{ij}] = \binom{m}{2} \cdot \frac{1}{365} \end{aligned}$$

# Rotating the table

$n$  people are sitting around a circular table. There is a name tag in each place. Nobody is sitting in front of their own name tag.

Rotate the table by a random number  $k$  of positions between 1 and  $n-1$  (equally likely)

Let  $X$  be the number of people that end up in front of their own name tag. Find  $\mathbb{E}[X]$ .

Decompose:

What  $X_i$  can we define that have the needed information?

LOE:

Conquer:

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Decompose: Define  $X_i$  as follows:

$$X_i = \begin{cases} 1 & \text{if person } i \text{ sits in front of their own name tag} \\ 0 & \text{otherwise} \end{cases}$$

Note:  $X = \sum_{i=1}^n X_i$

LOE:

$$\mathbb{E}[X] = \mathbb{E}[\sum_{i=1}^n X_i] = \sum_{i=1}^n \mathbb{E}[X_i]$$

Conquer:

These  $X_i$  are not independent!  
That's ok!!

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$X$  is the number of people that end up in front of their own name tag. **Find**  $\mathbb{E}[X]$ .

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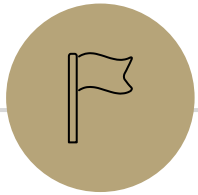
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LOE:

$$\mathbb{E}[X] = \mathbb{E}[\sum_{i=1}^n X_i] = \sum_{i=1}^n \mathbb{E}[X_i]$$

Conquer:

$$\mathbb{E}[X_i] = P(X_i = 1) = \frac{1}{n-1} \quad \mathbb{E}[X] = n \cdot \mathbb{E}[X_i] = \frac{n}{n-1}$$



## Extra Practice

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# Frogger



A frog starts on a 1-dimensional number line at 0.

Each second, independently, the frog takes a unit step right with probability  $p_1$ , to the left with probability  $p_2$ , and doesn't move with probability  $p_3$ , where  $p_1 + p_2 + p_3 = 1$ .

After 2 seconds, let  $X$  be the location of the frog. **Find  $\mathbb{E}[X]$ .**

# Frogger – Brute Force



A frog starts on a 1-dimensional number line at 0. At each second, independently, the frog takes a unit step right with probability  $p_R$ , to the left with probability  $p_L$ , and doesn't move with probability  $p_S$ , where  $p_L + p_R + p_S = 1$ . After 2 seconds, let  $X$  be the location of the frog. Find  $\mathbb{E}[X]$ .

We could find the PMF by computing the probability for each value in the range of  $X$ , and then applying definition of expectation:

$$p_X(x) = \begin{cases} p_L^2 & x = -2 \\ 2p_Lp_S & x = -1 \\ 2p_Lp_R + p_S^2 & x = 0 \\ 2p_Rp_S & x = 1 \\ p_R^2 & x = 2 \\ 0 & \text{otherwise} \end{cases}$$

*We think about the outcomes that correspond to each value of  $X$  and compute the probability of that. For example,  $X=0$  happens when the frog is at the same position after 2 sec – this means it either moved left and then right, or right and then left, or did not move both seconds.*

$$\mathbb{E}[X] = \sum_{\omega} P(\omega)X(\omega) = (-2)p_L^2 + (-1)2p_Lp_S + 0 \cdot (2p_Lp_R + p_S^2) + (1)2p_Rp_S + (2)p_R^2 = 2(p_R - p_L)$$

# Frogger – LOE



Or we can apply LoE!

A frog starts on a 1-dimensional number line at 0. At each second, independently, the frog takes a unit step right with probability  $p_R$ , to the left with probability  $p_L$ , and doesn't move with probability  $p_S$ , where  $p_L + p_R + p_S = 1$ . After 2 seconds, let  $X$  be the location of the frog. Find  $\mathbb{E}[X]$ .

Define  $X_i$  as follows:

$$X_i = \begin{cases} -1 & \text{if the frog moved left on the } i\text{th step} \\ 0 & \text{otherwise} \\ 1 & \text{if the frog moved right on the } i\text{th step} \end{cases}$$

$$\mathbb{E}[X_i] = -1 \cdot p_L + 1 \cdot p_R + 0 \cdot p_S = (p_R - p_L)$$

By Linearity of Expectation,

$$\mathbb{E}[X] = \mathbb{E}\left[\sum_{i=1}^2 X_i\right] = \sum_{i=1}^2 \mathbb{E}[X_i] = 2(p_R - p_L)$$

# Frogger – LOE



If we interested in a whole minute (60 sec), the first approach would be awful because we would need to compute many probabilities or deal with a gnarly summation! Instead, we can use LoE!

A frog starts on a 1-dimensional number line at 0. At each second, independently, the frog takes a unit step right with probability  $p_R$ , to the left with probability  $p_L$ , and doesn't move with probability  $p_S$ , where  $p_L + p_R + p_S = 1$ . After 60 seconds, let  $X$  be the location of the frog. Find  $\mathbb{E}[X]$ .

Define  $X_i$  as follows:

$$X_i = \begin{cases} -1 & \text{if the frog moved left on the } i\text{th step} \\ 0 & \text{otherwise} \\ 1 & \text{if the frog moved right on the } i\text{th step} \end{cases}$$

$$\mathbb{E}[X_i] = -1 \cdot p_L + 1 \cdot p_R + 0 \cdot p_S = (p_R - p_L)$$

By Linearity of Expectation,

$$\mathbb{E}[X] = \mathbb{E}\left[\sum_{i=1}^{60} X_i\right] = \sum_{i=1}^{60} \mathbb{E}[X_i] = 60(p_R - p_L)$$