Linearity of Expectation - Proof

Linearity of Expectation

For any two random variables X and Y:

$$\mathbb{E}[\boldsymbol{X}+\boldsymbol{Y}] = \mathbb{E}[\boldsymbol{X}] + \mathbb{E}[\boldsymbol{Y}]$$

Note: X and Y do not have to be independent

Proof:

$$\begin{aligned}
\mathbb{E}[X+Y] &= \Sigma_{\omega\in\Omega}\mathbb{P}(\omega)\big(X(\omega)+Y(\omega)\big) \\
&= \Sigma_{\omega\in\Omega}\mathbb{P}(\omega)X(\omega)+\mathbb{P}(\omega)Y(\omega) \\
&= \Sigma_{\omega\in\Omega}\mathbb{P}(\omega)X(\omega)+\Sigma_{\omega\in\Omega}\mathbb{P}(\omega)Y(\omega) \\
&= \mathbb{E}[X]+\mathbb{E}[Y]
\end{aligned}$$

37

Indicator Random Variables

For any event A, we can define the indicator random variable $\mathbf{1}[A]$ for A

$$\mathbf{1}[A] = \begin{cases} 1 & \text{if event A occurs} \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbb{P}(X=1) = \mathbb{P}(A)$$
$$\mathbb{P}(X=0) = 1-\mathbb{P}(A)$$

$$p_X(x) = \begin{cases} \mathbb{P}(A) & \text{if } x=1 \\ 1-\mathbb{P}(A) & \text{if } x=0 \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned}
&\mathbb{E}\,[X] \\
&= 1\cdot p_X(1)+0\cdot p_X(0) \\
&= p_X(1) = \mathbb{P}(A)
\end{aligned}$$

You'll also see notation like:

$\mathbb{1}[A]$, 1_A, $\mathbb{1}[\text{some boolean}]$

20

Pairs with the same birthday

In a class of m students, on average how many pairs of people have the same birthday?

Decompose:

LOE:

Conquer:

Rotating the table

n people are sitting around a circular table. There is a name tag in each place. Nobody is sitting in front of their own name tag.

Rotate the table by a random number k of positions between 1 and n-1 (equally likely)

Let X be the number of people that end up in front of their own name tag. Find $\mathbb{E}[X]$.

Decompose:

What X_i can we define that have the needed information?

LOE:

Conquer: