Example

How many length 5 strings over the alphabet $\{a, b, c, \ldots, z\}$ contain:
Exactly 2 'a's OR
Exactly 1 'b' OR
No 'x's

$|A| = \binom{5}{2} \cdot 25^3$

$|B| = \binom{5}{1} \cdot 25^4$

$|C| = 25^5$

$|A \cap B| = \binom{5}{2} \cdot \binom{3}{1} \cdot 24^2$

$|A \cap C| = \binom{5}{2} \cdot 24^3$

$|B \cap C| = \binom{5}{1} \cdot 24^4$

$|A \cap B \cap C| = \binom{5}{2} \cdot \binom{3}{1} \cdot 23^2$

$$
\begin{aligned}
|A \cup B \cup C| &= \\
&= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \\
&= \binom{5}{2} \cdot 25^3 + \binom{5}{1} \cdot 25^4 + 25^5 - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \\
&= 11{,}875{,}000 - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \\
&= 11{,}875{,}000 - \binom{5}{2} \cdot \binom{3}{1} \cdot 24^2 - \binom{5}{2} \cdot 24^3 - \binom{5}{1} \cdot 24^4 + |A \cap B \cap C| \\
&= 11{,}875{,}000 - 1{,}814{,}400 + |A \cap B \cap C| \\
&= 10{,}060{,}600 + |A \cap B \cap C| \\
&= 10{,}060{,}600 + \binom{5}{2} \cdot \binom{3}{1} \cdot 23^2 \\
&= 10{,}060{,}600 + 15{,}870 \\
&= 10{,}076{,}470
\end{aligned}
$$

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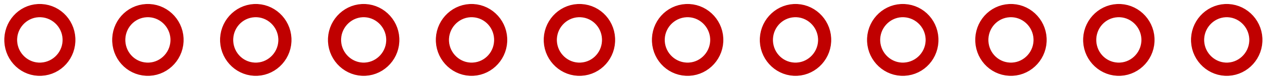
One More Counting Rule

You're going to buy one-dozen donuts (i.e., 12 donuts)

There are chocolate, strawberry, coconut, blueberry, and lemon (i.e. five types)

How many different donut boxes can you buy?

Consider two boxes the same if they contain the same number of every kind of donut (order doesn't matter)



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We've seen lots of ways to count

Sum rule (split into disjoint sets)

Product rule (use a sequential process)

Combinations (order doesn't matter)

Permutations (order does matter)

Principle of Inclusion-Exclusion

Complementary Counting

"Stars and Bars" $\binom{n+k-1}{k-1}$

Niche Rules (useful in very specific circumstances)

Binomial Theorem

Pigeonhole Principle

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Which Tool Do I Use?

Pick k things from universe of n ($n \geq k$)	Repetition is NOT allowed	Repetition IS allowed
Order does NOT matter	Combinations $\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$	Stars and Bars $\binom{n+k-1}{n-1}$ Be careful which is n and which is k. This is k donuts from n flavors.
Order does matter	Permutations $P(n,k) = \frac{n!}{(n-k)!}$	Product rule $n \cdot n \cdots n = n^k$

This is NOT foolproof! Sometimes you need a twist on the formula; sometimes it's a completely different tool. But a sign where to start.

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