CSE 312 : Quiz 4 Practice 2 Solutions

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Instructions

- You have twenty minutes to complete this exam.
- You are permitted one piece of 8.5x11 inch paper with handwritten notes (notes are allowed on both sides of the paper). You should also get a provided formula sheet (in section it'll be on different colored paper separate from the exam; if you take the exam with DRS it will be the last page of your exam).
- You may not use a calculator or any other electronic devices during the exam.
- We will be scanning your exams before grading them. Please write legibly, and avoid writing up to the edge of the paper.
- If you run out of room, you may also use the last page for extra space, but tell us where to find your answer if it's not right below the problem.
- Since you don't have a calculator, you are generally free to **not** simplify expressions (though you may if you think it will be helpful).
- In general, you should show us the work you used to get to an answer, and explanations will help us reward partial credit, but we do **not** expect explanations at the level we usually require on homeworks.

Advice

- Writing a few words about where an expression came from is often very helpful for awarding partial credit.
- Remember to take deep breaths.

Question	**Max points**
PDF/CDF	20
CLT	14
Grading Morale	1
Total	**35**

1. CLT

Suppose I have a flashlight which requires one battery to operate, and I have 18 identical batteries. I want to go camping for a week ($24 \times 7 = 168$ hours). If the lifetime of a single battery is a random variable distributed as an Exp(0.1)? **Estimate** this probability using the Central Limit Theorem. Do not compute it exactly.

You SHOULD NOT lookup values in the z-table for this problem. Instead, your solution should be an expression with Φ that can be evaluated using the lookup table (i.e. all inputs to Φ are non-negative) and a calculator (i.e. the input to Φ does not need to be simplified).

Solution:

The total lifetime of the battery is

$$X = X_1 + \cdots + X_{18}$$

where each $X_i \sim Exp(0.1)$ has

$$\mathbb{E}[X_i] = \frac{1}{0.1} = 10, \quad \text{and} \quad \text{Var}(X_i) = \frac{1}{0.1^2} = 100.$$

Hence,

$$\mathbb{E}[X] = 180, \quad \text{and} \quad \text{Var}(X) = 1800$$

by linearity of expectation and since variance adds for independent random variables.

By the Central Limit Theorem (CLT),

$$X \approx \mathcal{N}(\mu = 180, \sigma^2 = 1800),$$

so

$$\begin{aligned}
\mathbb{P}(X \geq 168) &\approx \mathbb{P}(\mathcal{N}(180, 1800) \geq 168) \\
&= \mathbb{P}\left(Z \geq \frac{168 - 180}{\sqrt{1800}}\right) \\
&= 1 - \Phi\left(\frac{168 - 180}{\sqrt{1800}}\right) \\
&= \Phi\left(\frac{168 - 180}{\sqrt{1800}}\right)
\end{aligned}$$

Note that we don't use the continuity correction here because the random variables we are summing are already continuous.

This input to $\Phi()$ is negative, so we'd really need to find $1 - \Phi\left(-1 \cdot \frac{168-180}{\sqrt{1800}}\right)$, though you might not notice that without a caclulator.

2. Bounds

Your Dungeons & Dragons character just got a new sword. Happily, it does a lot of damage; sadly, it is cursed.

Each time you attack using the sword, you roll a (fair) 20-sided die (numbered $1, 2, 3, ..., 20$). If you roll a 1, the sword's curse corrupts your character's moral standing a bit. If that happens 5 times (total), your character becomes evil.

Let X be the number of times you attack using the sword up to (and including) the one that makes your character evil.

(a) What is $\mathbb{E}[X]$? **Hint:** linearity
Solution:

This is the sum of geometrics (equivalently, a negative binomial); the expectation of one geometric is $1/(1/20) = 20$. Applying linearity, we get $5 \cdot 20 = 100$.

(b) What is Var (X)? **Hint:** The time between the i^{th} and $(i+1)^{\text{st}}$ corruptions are independent. **Solution:**

This is the sum of **independent** geometrics, so we can add together variances: the variance for one geometric is $\frac{1-1/20}{(1/20)^2}$

You may use E to refer to $\mathbb{E}[X]$ and V to refer to the variance of X.

(c) Use Markov's inequality to bound the probability that $X \geq 300$.

Solution:

Directly applying Markov: $\mathbb{P}(X \geq 300) \leq \frac{\mathbb{E}[X]}{300} = \frac{100}{300} = \frac{1}{3}$.

(d) The GM (who is designing the game) has decided your character is too strong with the sword, and wants to stop you from using it (or at least make you turn evil, that would be good drama). They want the probability that you **don't** get corrupted to be less than 1% (assuming you continue using the sword). Can they use each of these bounds to find a number of attacks (n) that will give a 99% chance of turning evil?

- Can you use Chebyshev's inequality? Select ALL that apply.
  ☐ Yes, you can use Chebyshev to find an n.
  ☐ No, Chebyshev is two-sided, so it cannot be used here at all.
  ☐ No, Chebyshev can only be applied "forward" (from an event, bound a probability), not "backward" (from a probability bound, find the event).
  ☐ No, the variance would change if you change n so Chebyshev would not be useful in this way.

  Solution:

  Yes, you can use Chebyshev to find an n. Chebyshev being two-sided will make the bound looser (i.e., bigger than it might be if you used a different bound), but you certainly can find an n this way.

- Can you use the Chernoff bound? Select ALL that apply.
  ☐ Yes, you can use Chernoff to find an n.
  ☐ No, Chernoff is two-sided, so it cannot be used here at all.
  ☐ No, Chernoff can only be applied "forward" (from an event, bound a probability), not "backward" (from a probability bound, find the event).
  ☐ No, Chernoff applies to sums of Bernoullis; this is the sum of geometrics.

  Solution:

  No, Chernoff applies to sums of Bernoullis; this is a sum of geometrics.

  (The other two no answers are inaccurate).