CSE 312 : Quiz 3 Practice 2 Solutions

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Instructions

- You have twenty minutes to complete this exam.
- You are permitted one piece of 8.5x11 inch paper with handwritten notes (notes are allowed on both sides of the paper). You should also get a provided formula sheet (in section it'll be on different colored paper separate from the exam; if you take the exam with DRS it will be the last page of your exam).
- You may not use a calculator or any other electronic devices during the exam.
- We will be scanning your exams before grading them. Please write legibly, and avoid writing up to the edge of the paper.
- If you run out of room, you may also use the last page for extra space, but tell us where to find your answer if it's not right below the problem.
- Since you don't have a calculator, you are generally free to **not** simplify expressions (though you may if you think it will be helpful).
- In general, you should show us the work you used to get to an answer, and explanations will help us reward partial credit, but we do **not** expect explanations at the level we usually require on homeworks.

Advice

- Writing a few words about where an expression came from is often very helpful for awarding partial credit.
- Remember to take deep breaths.

Question	Max points
PDF/CDF	20
CLT	14
Grading Morale	1
Total	**35**

1. PDFs and CDFs

(a) Let $f_X(x) = \begin{cases} \frac{1}{4}(x+1) & \text{for } 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$

Compute $\mathbb{P}(1 < X < 2)$.

Solution:

We compute:

$$\mathbb{P}(1 < X < 2) = \int_1^2 \frac{1}{4}(x+1)\,dx.$$

Evaluating:

$$\frac{1}{4}\int_1^2 x+1\,dx = \frac{1}{4}\left[\frac{x^2}{2}+x\right]_1^2 = \frac{1}{4}(4-\frac{3}{2}) = \frac{5}{8}$$

Thus,

$$\mathbb{P}(1 < X < 2) = \frac{5}{8}.$$

(b) Let $F_X(x)$ be the CDF of a continuous random variable X. Suppose $a < b < c < d$. Express

$$\mathbb{P}(X < a \cup b < X < c \cup X > d)$$

in terms of $F_X(x)$.

Solution:

The probability of X being in the interval (b, c) is:

$$\mathbb{P}(b < X < c) = F_X(c) - F_X(b).$$

The probability of X being less than a is simply:

$$\mathbb{P}(X < a) = F_X(a).$$

The probability of X being greater than d is:

$$\mathbb{P}(X > d) = 1 - F_X(d).$$

Since these events are disjoint, their total probability is:

$$\mathbb{P}(X < a \text{ or } b < X < c \text{ or } X > d) = F_X(a) + (F_X(c) - F_X(b)) + (1 - F_X(d)).$$

Simplifying:

$$1 + F_X(a) + F_X(c) - F_X(b) - F_X(d).$$

(c) Determine the constant c that makes the following function a valid PDF:

$$f_X(x) = \begin{cases} ce^{-3x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Note: Recall that $\lim_{x\to\infty} e^{-x} = 0$.

Solution:

The function must satisfy:

$$\int_0^\infty ce^{-3x}\,dx = 1.$$

Evaluating the integral:

$$c\int_0^\infty e^{-3x}\,dx = c\left[\frac{e^{-3x}}{-3}\right]_0^\infty.$$

Using the given fact that $e^{-\infty} = 0$, we compute:

$$c\left(0 - \left(-\frac{1}{3}\right)\right) = c \cdot \frac{1}{3}.$$

Setting this equal to 1:

$$\frac{c}{3} = 1 \quad \Rightarrow \quad c = 3.$$

(d) Find the PDF for the following CDF. Be sure to include all cases.

$$F_X(x) = \begin{cases} 0 & \text{for } x < 0 \\ \frac{x^3}{18} & \text{for } 0 \leq x < 2 \\ 7 + \frac{5(x-2)^2}{9} & \text{for } 2 \leq x < 3 \\ 1 & \text{for } x \geq 3 \end{cases}$$

Solution:

Differentiate each component in the piecewise of the CDF.

$$f_X(x) = \begin{cases} 0 & \text{for } x < 0,\, x \geq 3 \\ \frac{x^2}{6} & \text{for } 0 \leq x < 2 \\ \frac{10(x-2)}{9} & \text{for } 2 \leq x < 3 \end{cases}$$

(e) Find the expectation of a random variable Y given by the PDF $f_Y(y) = \begin{cases} \frac{3}{8}y^2 & \text{for } 0 \leq y \leq 2 \\ 0 & \text{otherwise} \end{cases}$

Solution:

$$\mathbb{E}[Y] = \int_0^2 y \cdot \frac{3}{8} y^2 dy = \frac{3}{8} \int_0^2 y^3 dy = \frac{3}{8} \left[\frac{1}{4} y^4 \right]_0^2 = \frac{3}{8}(4 - 0) = \frac{3}{2}$$