CSE 312 : Winter 2025 Midterm Exam, Form B Solutions

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Instructions

- You have ninety minutes to complete this exam.
- You are permitted one piece of 8.5x11 inch paper with handwritten notes (notes are allowed on both sides of the paper).
- You may not use a calculator or any other electronic devices during the exam.
- We will be scanning your exams before grading them. Please write legibly, and avoid writing up to the edge of the paper.
- If you run out of room, you may also use the last page for extra space, but tell us where to find your answer if it's not right below the problem.
- In general, show us the work you used to get to an answer; explanations will help us award partial credit, but we do not expect explanations at the level we usually require on homeworks.

Simplification Expectations

- Since you don't have a calculator for this exam, you do not have to do simplifications that could be done easily with a calculator. For example
$$\frac{\binom{5}{3} \cdot 17^2}{1-p} + 5^3$$
can be given as a final answer.
- However, answers which are much more complicated than the expected answer may receive deductions. For example: $\sum_{i=0}^{n} \binom{n}{i}$ or $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}$ are **not** simplified sufficiently.

Advice

- Writing a few words about where an expression came from is often very helpful for awarding partial credit.
- Remember to take deep breaths.

Question	Max points
Counting	12
Probability	16
Conditioning	18
Random Variable	18
Multiple Choice	15
Grading Morale	1
Total	**80**

1. Frost Byte [12 points]

Frost Byte is a beloved ice cream shop at UW with its **10** distinct flavors of ice cream, including chocolate, cookies and cream, strawberry, etc.

You *love* ice cream, so you will be ordering many scoops of ice cream.

(a) You intend to order a mega-sampler—20 scoops of ice cream. Frost Byte allows customers to order multiple scoops of the same flavor. For example, you could order 20 scoops of chocolate if you wanted to! If you order **20 scoops**, how many total mega-samplers are possible? **Note:** The order in which the scoops are chosen does not matter. [4 points]

Solution:

This is a classic **stars and bars** problem. The number of combinations is given by:

$$\binom{n+k-1}{k-1},$$

where $n = 20$ (scoops) and $k = 10$ (flavors). Thus, the total number of combinations is:

$$\binom{20+10-1}{10-1} = \binom{29}{9}.$$

(b) One day, Frost Byte is running low on supply, so they enforce a rule: customers can order **at most 1 scoop per flavor**. You decide to buy a medium-sampler, a set of 6 scoops. How many medium-samplers are there where **at least one scoop of chocolate or vanilla** is included? [4 points]

Solution:

We can solve this problem with **complementary counting**. First, calculate the total number of ways to choose 6 scoops without any restrictions:

$$\binom{10}{6}.$$

Next, subtract the number of ways that **neither chocolate nor vanilla** is included. Since there are $10 - 2$ flavors left (excluding chocolate and vanilla), the number of invalid combinations is:

$$\binom{10-2}{6}.$$

Therefore, the number of valid combinations is:

$$\binom{10}{6} - \binom{10-2}{6}.$$

We can also solve this using the **Principle of Inclusion-Exclusion (PIE)**. Let:

- A = combinations with at least one scoop of chocolate,
- B = combinations with at least one scoop of vanilla.

The number of valid combinations is:

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

- $|A| = \binom{10-1}{6-1}$ (choose the remaining $6 - 1$ scoops from the other $10 - 1$ flavors),
- $|B| = \binom{10-1}{6-1}$ (similarly for vanilla),
- $|A \cap B| = \binom{10-2}{6-2}$ (both chocolate and vanilla are included).

Thus, the total number of valid combinations is:

$$2 \cdot \binom{10-1}{6-1} - \binom{10-2}{6-2}.$$

(c) At the end of the day, Frost Byte closes, and the workers decide to take home some of the leftover ice cream. There is one open container for each of the **10** flavors that the employees take home. The three workers—Michelle, Henry, and Sarah—agree to split the flavors among themselves. Michelle will take 4 flavors, Henry will take 3 flavors, and Sarah will take 3 flavors. How many ways can the 10 open containers be distributed among Michelle, Henry, and Sarah? [4 points]

Solution:

This is a **multinomial coefficient** problem. The number of ways to distribute the flavors is:

$$\binom{10}{4, 3, 3} = \frac{10!}{4! \cdot 3! \cdot 3!}.$$

2. Cafe Chance [16 points]

You're a barista working at Cafe Chance! Through many hours of working, you've realized that the people who visit your cafe are extremely predictable. Any student that visits your cafe has a 60% chance of ordering a latte, a 25% chance of ordering a mocha, and a 15% chance of ordering a cappuccino (independently of everyone else).

Today on your shift, you see a group of 6 students enter the cafe at the same time and prepare to take their order.

(a) What is the probability that at least two of the 6 students order a latte? [4 points]

Solution:

Apply complementary counting on no student or exactly one student ordering a latte.

Case 1: Find the probability that no student orders a latte. Find the complement of ordering a latte, and apply to all 6 students.

$$(1-0.60)^6$$

Case 2: Find the probability that exactly one student orders a latte.

$$\binom{6}{1} \cdot 0.60 \cdot (1-0.60)^5$$

Apply complement

$$1 - \left((1-0.60)^6 + \binom{6}{1} \cdot 0.60 \cdot (1-0.60)^5\right)$$

(b) What is the probability that exactly one of the 6 students orders a mocha and exactly two order a cappuccino? [4 points]

Solution:

Use combinations to count number of possibilities of one student ordering a mocha and two students ordering cappuccinos. The remaining two students must order lattes.

$$\binom{6}{1} \cdot 0.25 \cdot \binom{6-1}{2} \cdot 0.15^2 \cdot 1^3 \cdot 0.60$$

(c) What is the probability that the second and the fifth student order the same drink? [4 points]

Solution:

There are three cases for this question, that the students both order a latte, a mocha, or a cappuccino.

$$0.60^2 + 0.25^2 + 0.15^2$$

(d) Suppose it turns out that 3 (identical) lattes, 2 (identical) mochas, and 1 cappuccino are ordered. You place them all on the counter, and each person grabs a (not-yet-taken) drink (uniformly) at random. What is the probability that each person gets a correct drink? [4 points]

Solution:

$$\frac{3!2!1!}{6!}$$

3. Expecto Patronum! [18 points]

At Hogwarts, Professor Dumbledore can detect students moving within the castle after curfew using either the Intruder Charm or Marauders Map. If a student is detected by either the Intruder Charm or Marauders Map (or both), they will alert Fluffy, the giant three-headed dog.

If there is a student moving within the corridors, the Intruder Charm will detect it with probability 0.85, and Marauders Map will detect it with probability 0.9.
If there is no student moving, the Intruder Charm will falsely detect anyway with probability 0.2, and Marauders Map will falsely detect anyway with probability 0.1.
Based on past history, the probability that there is a student moving is 0.75.

Assume these defenses have proprietary algorithms, so that conditioned on there being a student moving (or not), the events of detecting (or not) for each defense is independent.

Define

- S to be the event that there being a student moving,
- I to be the event that the Intruder Charm detects
- M to be the event that Marauders Map detects.

(a) Fill in the blank to show the notation for this probability (using the events defined above): Given there is a student moving and the Marauders Map does not detect, what is the probability that the Intruder Charm detects a student moving? [2 points]
(Do not attempt to simplify, write exactly the notation that corresponds to the English description).

Solution:

$P(I \mid S, M^C)$

$\mathbb{P}$ (________________)

(b) Compute the probability in the last part (i.e., given there is a student moving and the Marauders Map does not detect, what is the probability that the Intruder Charm detects a student moving?) [4 points]

Solution:

We need to find $P(I|S, M^C)$. Since the events of detecting motion for each defense is independent, by the definition of conditional independence, $P(I|S, M^C) = P(I|S)$ since events M and I are conditionally independent given event S. So we need to find the probability of an Intruder Charm detecting movement given there is movement outside, which is 0.85, which is $P(I|S)$. Thus the probability that the Marauders Map detects movement given there is movement outside and that the Intruder Charm does not detect motion is 0.85.

(c) What is the probability that there is no student moving but Fluffy is alerted anyway(i.e. the Intruder Charm or Marauders Map or both detect movement)? [4 points]

Solution:

We want to find $P(I \cup M|S^c)$. Using the Principle of Inclusion-Exclusion rule, we want to find $P(I \cup M|S^c) = P(I|S^c) + P(M|S^c) - P(I \cap M|S^c)$. Simplifying it even more using the definition of conditional probability, $P(I \cup M|S^c) = P(I|S^c) + P(M|S^c) - P(I|S^c) \cdot P(M|S^c)$. We know from the problem that the probability I detecting motion if there is no movement is 0.2, and the probability of M detecting motion when there is no movement is 0.1. So substituting those values for $P(I|S^c) = 0.2$ and $P(M|S^c) = 0.1$, we get $P(I \cup M|S^c) = 0.2 + 0.1 - 0.2 \cdot 0.1 = 0.28$. Thus, given that there is no moving object, the probability that the defenses alerts Fluffy is

$$P(I \cup M|S^c) = 0.28$$

(d) What is the probability both the Intruder Charm and Marauders Map detect? (Hint: LTP) [4 points]

Solution:

We need to find $P(I, M)$. Using the law of total probability, there are only two cases for $P(I, M)$ one is if there is movement, the other if there is no movement. so we want to find $P(I, M|S) = P(I, M|S)P(S) + P(I, M|S^c)P(S^c)$. We know $P(I, M|S) = P(I|S) \cdot P(M|S)$ and $P(I, M|S^c) = P(I|S^c) \cdot P(M|S^c)$. From the info in the problem we get $P(I, M|S^c) = 0.2 \cdot 0.1$ and $P(I, M|S) = 0.85 \cdot 0.9$. Additionally, we know $P(S) = 0.75$ and $P(S^c) = 0.25$. So, $P(I, M) = 0.85 \cdot 0.9 \cdot 0.75 + 0.2 \cdot 0.1 \cdot 0.25 = 0.57875$. Thus the probability the Intruder Charm and Marauders Map detects movement is 0.57875.

(e) What is the probability that there is a student moving given that both defenses detect? You can use the answer to previous parts in your expression (e.g. refer to the correct answer from part (d) as d) [4 points]

Solution:

We want to find the probability that there is a moving object given that both defenses detect motion. Thus we want to find $P(S|I, M)$. Using Bayes theorem we get $P(S|I, M) = \dfrac{P(I, M|S)P(S)}{P(I, M)}$. We know from the previous parts that by the definition of conditional independence, $P(I \cap M|S)$ or $P(I, M|S)$ $= P(I|S) \cdot P(M|S)$, thus $P(I, M|S) = 0.85 \cdot 0.9 = 0.765$. We also know from the problem description that $P(S) = 0.75$, so $P(I, M|S)P(S) = 0.765 \cdot 0.75 = 0.57375$. From part d we know $P(I, X)$. Thus $P(X|I, M) = \dfrac{0.85 \cdot 0.9 \cdot 0.75}{0.85 \cdot 0.9 \cdot 0.75 + 0.2 \cdot 0.1 \cdot 0.25} \approx 0.99136$. Thus the probability there is movement outside given both I and M detect movement is

$$0.99136$$

4. Gift Exchange [18 points]

A group of 4 friends is doing a gift exchange, where each person will be secretly assigned another member of the group for whom they buy a present. The friends will each put their own name into a hat, and then draw one name each to pick whom they give their gift to. We will analyze the following method of drawing names.

Method 1: Each friend picks a name out of the ones remaining in the hat uniformly at random (without looking). Once everyone has picked, they read the name of their recipient. (They do not place the name the pick back into the hat, so each name is given to exactly one person).

This method ensures every assignment of names is equally likely, but some people might get their own name. Let X be the number of people that **do** get their own name.

(a) What is the probability that person i **does** get their own name? [2 points]

Solution:

Let N be the event that person i gets their own name. Then, we are looking for

$$\mathbb{P}(N) = (\frac{1 \cdot 3!}{4!}) = \frac{1}{4}$$

where the equation for $\mathbb{P}(N)$ comes from the fact that every assignment of names is equally likely.

(b) Calculate $\mathbb{E}[X]$, or the expected number of people that get their own name. [4 points]
Hint: Use Linearity of Expectation. You can use the answer to part (a) by referring to the variable a. **Solution:**

Define the random variable X_i which is 1 if person i gets their own name and 0 otherwise. Then, $X = X_1 + X_2 + X_3 + X_4$. In part (a), we calculated $\mathbb{E}[X_i] = \mathbb{P}(X_i = 1) = \frac{1}{4}$. Therefore, using LOE,

$$\mathbb{E}[X] = 4a = 4(\frac{1}{4}) = 1$$

We call the result of Method 1 a *valid assignment* if each person draws a name that is not their own. There are 9 possible valid assignments for the 4 friends.

Suppose we repeat Method 1 until we get a valid assignment, where each round is independent of the last. Let Y denote the number of rounds until this is achieved. So, for example, $Y = 1$ means we get a valid assignment on the first try. $Y = 2$ means someone got their own name on the first draw, so all names were placed back into the hat and no one got their own name the second time.

(c) Calculate $\mathbb{E}[Y]$, or the expected number of rounds needed. [4 points]
Hint: Y is a random variable in the discrete zoo, so your answer can be a single number. However, we will also accept answers with a summation sign with constants plugged in for full credit.

Solution:

From PMF: For each round, there is a $\frac{9}{4!}$ chance of getting a valid assignment, and a $1 - \frac{9}{4!}$ chance of not getting a valid assignment. Therefore, the PMF looks like

$$p_Y(y) = \begin{cases} (1 - \frac{9}{4!})^{y-1}(\frac{9}{4!}) & y \in \mathbb{Z}^+ \\ 0 & \text{otherwise} \end{cases}$$

Using this,

$$\mathbb{E}[Y] = \sum_{i=1}^{\infty} i(1 - \frac{9}{4!})^{i-1}(\frac{9}{4!})$$

Using RV Zoo: This is a geometric random variable with $p = \frac{9}{4!}$, so its expectation is

$$\mathbb{E}[Y] = \frac{1}{\frac{9}{4!}} = \frac{4!}{9} = \frac{8}{3}$$

We say a valid assignment has a *pair* if two people are buying for each other. Assuming we get a valid assignment using Method 1, let Z denote the number of pairs in the assignment.

(d) Write the PMF for Z. Be sure to include all cases. [4 points]
Hint: The PMF for Z sums to 1. Try to find the probability for one case and use that to fill in the PMF.

Solution:

There are either 0 pairs or 2 pairs. There cannot be 1 pair since we have a valid assignment (e.g. the remaining two people must have each others names). Therefore, $\Omega_Z = \{0, 2\}$.

We will start by finding $p_Z(2)$. There are $\binom{4}{2}$ possible ways of making the first pair, and the remaining pair is then fixed. However, this overcounts by a factor of 2 since the first and second pair are interchangeable. Therefore, $p_Z(2) = \frac{1}{9}(\frac{\binom{4}{2}}{2}) = \frac{1}{3}$.

Since the values in the PMF sum to 1, we know $p_Z(0) = 1 - p_Z(2)$. Therefore,

$$p_Z(z) = \begin{cases} 1 - \frac{1}{3} & z = 0 \\ \frac{1}{3} & z = 2 \\ 0 & \text{otherwise} \end{cases}$$

(e) Let A be a random variable which is 4 with probability $1/4$, and $8/3$ with probability $3/4$. Note that $\mathbb{E}[A] = 3$
Give a formula for the variance of A. [4 points]
You do not need to fully simplify, but your expression should use only constants, and not need a summation symbol (Σ) or ellipses (...). (But you may use $+, \cdot$, etc.) **Solution:**

We know Var$(A) = \mathbb{E}[A^2] - \mathbb{E}[A]^2$.

$$\mathbb{E}[A] = 4 \cdot \frac{1}{4} + \frac{8}{3} \cdot \frac{3}{4} = 3$$

$$\mathbb{E}[A^2] = 16 \cdot \frac{1}{4} + \frac{64}{9} \cdot \frac{3}{4} = 4 + \frac{16}{3} = \frac{28}{3}$$

Therefore,

$$\text{Var}(A) = \frac{28}{3} - 3^2 = \frac{28}{3} - \frac{27}{3} = \frac{1}{3}$$

5. Multiple Choice [15 points]

For the questions below,

- Questions with ◯ Circles have exactly one best answer. **Fully fill in** the circle for the one best answer.
- Questions with ☐ squares are "mark all that apply" questions. **Fully fill in** the square for all correct options.

(a) Suppose we have a set of n people which we want to split into two groups of size k and size $n-k$, respectively. Which of the following correctly counts the number of options? **Mark ALL that apply**:

☐ $\binom{n}{k} \cdot \binom{n}{n-k}$
☐ $\binom{n}{k}$
☐ $\binom{n}{n-k}$
☐ None of the above.

Solution:

Options 1 and 3 are correct.

(b) For random variables X and Y when can we say $\mathbb{E}[(X+Y)^2] = \mathbb{E}[X^2] + 2\mathbb{E}[XY] + \mathbb{E}[Y^2]$.

◯ The identity is always true.
◯ The identity is sometimes true—specifically when X, Y are independent.
◯ The identity is sometimes true—specifically when X, Y are the same type of distribution (e.g., both binomial or both geometric).
◯ The identity is never true.

Solution:

Always

(c) You are rolling a fair 6-sided-die, hoping to see as many 3's as possible. You roll the die 10 times (independently), how many 3's do you expect to see on average?

◯ $\frac{6}{10}$
◯ $\frac{10}{6}$
◯ $10 \cdot 6$
◯ $10 \cdot \frac{1}{6} \cdot \frac{5}{6}$ **Solution:**

10/6 (you might know the expectation of a binomial, or you can use linearity of expectation).

(d) Which of the following is a correct variant of the complementation law?

◯ $\mathbb{P}(A|B) = 1 - \mathbb{P}\left(A \mid \overline{B}\right)$
◯ $\mathbb{P}(A|B) = 1 - \mathbb{P}\left(\overline{A} \mid B\right)$
◯ $\mathbb{P}(A|B) = 1 - \mathbb{P}\left(\overline{A} \mid \overline{B}\right)$
◯ $\mathbb{P}(A|B) = 1 - \mathbb{P}\left(A \mid B\right)$

Solution:

$\mathbb{P}(A|B) = 1 - \mathbb{P}\left(\overline{A} \mid B\right)$

(e) Suppose you know $\text{Var}(X)$ and $\text{Var}(Y)$, what is the formula for $\text{Var}(aX + bY)$ (for constants a, b)

◯ $a \cdot \text{Var}(X) + b \cdot \text{Var}(Y)$

◯ $a^2 \cdot \text{Var}(X) + b^2 \cdot \text{Var}(Y)$

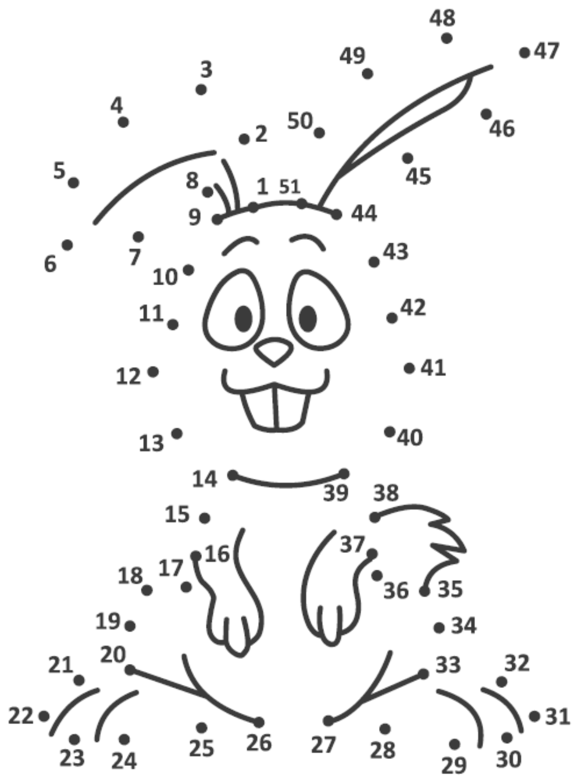
◯ $a + b + \text{Var}(X) + \text{Var}(Y)$

◯ There is not enough information to determine $\text{Var}(aX + bY)$ **Solution:**

Options 4 (if X, Y are dependent the variances may not simply add).

6. Grading Morale [1 point]

Put something on this page or finish the image below. Depending on your mood, this might be a poem, the statement of your favorite theorem, the statement of your least favorite theorem, or a piece of art. As long as you make some mark on this page, you will get the point. Looking at these helps keep the TAs happy while grading.



Additional space for any prior problem. Be sure to tell us to look here on the original problem!

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