CSE 312 : Winter 2025 Final Exam: Form A

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Instructions

- You have 110 minutes to complete this exam.
- You are permitted one piece of 8.5x11 inch paper with handwritten notes (notes on both sides allowed).
- You are also provided a reference sheet with the exam.
- You may not use a calculator or any other electronic devices during the exam.
- We will be scanning your exams before grading them. Please write legibly, and avoid writing up to the edge of the paper. If you run out of room, you may also use the last page for extra space, but tell us where to find your answer if it's not right below the problem.
- **Please put final answers in provided boxes on longer problems.**
- In general, show us the work you used to get to an answer; explanations will help us award partial credit, but we do not expect explanations at the level we usually require on homeworks.

Simplification Expectations

- Since you don't have a calculator for this exam, you do not have to do simplifications that could be done easily with a calculator. For example, the expression below is simplified enough to be a final answer.

$$\frac{\binom{5}{3} \cdot 17^2}{1-p} + 5^3$$

- However, answers which are much more complicated than the expected answer may receive deductions. For example: $\sum_{i=0}^{n} \binom{n}{i}$, or $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}$ are **not** simplified sufficiently. Generally summation notation or "..." are not sufficiently simple, but a few plus signs are fine.

Advice

- Writing a few words about where an expression came from is often very helpful for awarding partial credit.
- Remember to take deep breaths.

Question	**Max points**
Multiple Choice	27
Small Problems	10
I Show Speed (Random Variables)	16
It's Better Together (Multiple Variables)	10
Avada Kedavra! (Conditioning)	10
Potato, Potahto (Concentration)	16
Grogu Snacking (MLE)	10
Morale	1
Total	**100**

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1. [Multiple Choice] [27 points]

For the questions below,

- Questions with ◯ Circles have exactly one best answer. **Fully fill in** the circle for the one best answer.
- Questions with ☐ squares are "mark all that apply" questions. **Fully fill in** the square for all correct options.

(a) Suppose we have two coins.
Coin 1 shows up heads with probability 0.4.
Coin 2 shows up heads with probability 0.7.

(i) Suppose we flip Coin 1 twenty times independently. What is the probability of observing 12 heads followed by 8 tails (i.e., the first 12 flips are heads, and the last 8 are tails).

◯ $0.4^{12} \cdot 0.6^{8}$
◯ $\binom{20}{12} 0.4^{12} \cdot 0.6^{8}$
◯ $\binom{20}{8} 0.4^{12} \cdot 0.6^{8}$
◯ $0.4^{8} \cdot 0.6^{12}$

(ii) Suppose we choose Coin 1 with unknown probability θ (and Coin 2 with probability $(1-\theta)$) and flip that SAME coin at every trial independently, observing 12 heads followed by 8 tails. Which of the following is the likelihood of the data, given unknown parameter θ?

◯ $\theta^{12}(1-\theta)^{8}$
◯ $\theta^{8}(1-\theta)^{12}$
◯ $(0.4^{12} \cdot 0.6^{8})\theta + (0.7^{12} \cdot 0.3^{8})(1-\theta)$
◯ $(0.7^{12} \cdot 0.3^{8})\theta + (0.4^{12} \cdot 0.6^{8})(1-\theta)$

(b) Suppose X is a continuous RV with CDF $F_x()$ where we know the following values:
$F_x(2) = 0.12$
$F_x(3) = 0.21$
$F_x(4) = 0.30$
$F_x(5) = 0.45$
$F_x(6) = 0.60$

What is $\mathbb{P}(3 \leq X \leq 5)$?

◯ 0.67
◯ 0.24
◯ 0.96
◯ It can be computed with the information in the problem, but is not one of the numbers above.
◯ Unknown, because we do not know the PDF.

(c) If Cov$(X, Y) \neq 0$ for some random variables X and Y, then they are not independent.

◯ True
◯ False

(d) Knowing the joint distribution (joint PDF or PMF) of two RVs X, Y allows us to compute which of the following quantities (at least writing out the sums/integrals for them)? Select all that apply:

☐ $\mathbb{E}[X + Y]$
☐ Var$(X - Y)$
☐ $\mathbb{P}(X > Y)$
☐ Cov(X, Y)
☐ The marginal distributions of X and Y

(e) Suppose event A has probability 0.5 and B has probability 0.3, such that $P(A \cup B) = 0.65$. Then events A and B are independent.

◯ Always True
◯ Sometimes True
◯ Never True

(f) Let $X_1, X_2, ..., X_n$ be i.i.d. random variables, each with mean μ and variance σ^2. Which of the following accurately describe $X = \sum_{i=1}^{n} X_i$ (Mark ALL that apply)

☐ The Central Limit Theorem guarantees $X \sim \mathcal{N}(0, 1)$
☐ The Central Limit Theorem guarantees $X \sim \mathcal{N}()$, but not standardized
☐ $\mathbb{E}[X] = n\mu$
☐ Var$(X) = n\sigma^2$
☐ Var$(X) = n^2\sigma^2$

(g) For all random variables X and Y we have $\mathbb{E}[(X + Y)^2] = \mathbb{E}[X^2] + 2\mathbb{E}[XY] + \mathbb{E}[Y^2]$.

◯ Always True
◯ Sometimes True
◯ Never True

(h) Suppose you have a multidimensional Gaussian distribution, where $X_1 \sim \mathcal{N}(3, 5)$, $X_2 \sim \mathcal{N}(-1, 3)$ such that Cov$(X_1, X_2) = -1$. Fill in the covariance matrix

$$\Sigma = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$$

2. Small Questions [10 points]

(a) The city of Thneedville is mapped out on a 10x15 grid. Ted lives at (0,0) and needs to get to the city exit at (10,15) where he can exit the city to go meet the Once-ler. On the way, Ted must stop at Audrey's house at (4,5). Assume Ted can only move North (up) or East (right) by one step at a time. Find the number of possible paths he can take that go through (4,5). [3 points]

(b) Let $X \sim \mathcal{N}(5, 4)$. What is $\mathbb{P}(X \leq 1)$? You must give a number taken from the Φ-table on the reference sheet. [3 points]

(c) At the Sea world, a new exhibit consisting of 5 different species of fish and 5 different species of amphibians is to be formed from a pool of 10 fish species and 9 amphibian species. How many exhibits are possible if ...

(i) ... there are no additional restrictions on which species can be selected? [2 points]

(ii) ... 2 particular fish species cannot both be in the exhibit (e.g., they have a predator-prey relationship)? [2 points]

3. I Show Speed [Random Variables] [16 points]

In the small town of RVillage, an annual racing tournament is about to begin! Four skilled drivers—Alice, Bob, Charlie, and Daniel—have arrived with their custom-built cars, each with unique performance characteristics.

Drivers and Their Cars

- **Alice's Car**: Alice's car has a special engine that produces a speed of $10X + 35$ miles per hour, where X is determined by rolling a fair three-sided die (values $1, 2,$ or 3) just before the race begins.
- **Bob's Car**: Bob's car uses an experimental system that activates a random number of its 90 identical thruster units, each with a 50% chance of activating. The car's speed in miles per hour equals the number of activated thrusters, making it a Binomial$(90, 1/2)$ random variable.
- **Charlie's Car**: Charlie's car has a more consistent performance, with a speed that follows a Normal distribution with mean 50 mph and standard deviation $\sqrt{10}$ mph.
- **Daniel's Car**: Daniel's car's speed measured in miles per hour follows an exponential distribution with parameter $\lambda = 1/20$.

Once determined at the start of each race, each car maintains its speed consistently throughout the entire track. Let A, B, C, D be the speeds of the drivers' cars in order.

(a) What is the probability mass function (PMF) for Alice's car speed?

(b) Suppose Alice rolls a 2, what is the probability that Charlie's car is faster than Alice's car? That is, find $P(C > A|\text{Alice rolled a 2})$. You may leave expressions involving $\Phi()$ unsimplified in your computation.

Here are the key facts of the problem from the last page

- **Alice's Car**: Alice's car has a special engine that produces a speed of $10X + 35$ miles per hour, where X is determined by rolling a fair three-sided die (values $1, 2,$ or 3) just before the race begins.
- **Bob's Car**: Bob's car uses an experimental system that activates a random number of its 90 identical thruster units, each with a 50% chance of activating. The car's speed in miles per hour equals the number of activated thrusters, making it a Binomial$(90, 1/2)$ random variable.
- **Charlie's Car**: Charlie's car has a more consistent performance, with a speed that follows a Normal distribution with mean 50 mph and standard deviation $\sqrt{10}$ mph.
- **Daniel's Car**: Daniel's car's speed measured in miles per hour follows an exponential distribution with parameter $\lambda = 1/20$.

(c) What is the probability that the Daniel's car's speed is between 10 and 30 miles per hour? That is, find $P(10 < D < 30)$.

(d) The race officials decide to award a 'consistency prize' to the driver whose car has the lowest variance. Compute the variance of each car **Hint:** remember the formula sheet!

Alice: __________ Bob: __________ Charlie: __________ Daniel: __________

4. It's Better Together [10 points]

Let X and Y be continuous random variables with the following joint distribution.

$$f_{X,Y}(x,y) = \begin{cases} \frac{2x}{y^2} & x \in [0,1], y \in [\frac{1}{2}, 1] \\ 0 & \text{otherwise} \end{cases}$$

The marginal distribution for Y is

$$f_Y(y) = \begin{cases} \frac{1}{y^2} & y \in [\frac{1}{2}, 1] \\ 0 & \text{otherwise} \end{cases}$$

(a) Compute $f_X(x)$ [4 points].

(b) Compute $\mathbb{E}[X|Y=y]$, where $y \in [\frac{1}{2}, 1]$. [4 points].

(c) Write down an integral expression for $\mathbb{E}[XY^2]$. You should fill in the bounds and function we are integrating, but you do not need to evaluate the integral. [2 points].

$$\int_{\square}^{\square} \int_{\square}^{\square} \underline{\hspace{4cm}} \, d\square \, d\square$$

5. Avada Kedavra! [10 points]

After hearing many complaints from the students at Hogwarts about Fluffy being falsely alerted in the middle of the night, Professor Dumbledore decided to investigate a better solution for detecting students moving within the castle after curfew. Professor Dumbledore decides the best defense will be the more powerful Caterwauling Charm. However, before performing the spell, he wants to get a sense of why students are sneaking through the castle.

(a) Let C be the event "the Caterwauling Charm goes off" and S be the event "there is a student in the hallway." According to the The Standard Book of Spells Vol 5. Professor Dumbledore knows the following probabilities:

- Probability the Caterwauling Charm goes off given there is a student in the hallway, $\mathbb{P}(C|S)$, is 0.80.
- Probability the Caterwauling Charm goes off given there is no student in the hallway, $\mathbb{P}(C|S^c)$, is 0.02.
- Last year the probability that there is a student in the hallway at any given time, $\mathbb{P}(S)$, is 0.6.

Find the probability there is a student in the hallway given the Caterwauling Charm goes off. i.e., find $P(S|C)$

(b) Fred and George Weasley have been running an illegal butter beer stand at night (that's why so many students are out after curfew). Suppose every student who walks by their stand either buys a drink or doesn't (max 1 drink per student).
If Fluffy is awake on a given night, 12 students walk by their stand, and each buys a drink independently with probability 0.75. If Fluffy is asleep, 27 students walk by their stand, and each buys a drink independently with probability 0.9. The probability Fluffy is awake is 0.35. How many butter beers do they expect to sell on a given night?

(c) Suppose it costs 10 bronze knuts per night for the Weasley twins to operate the stand and one butter beer makes them 3 knuts. Let X be their total profit (sales minus operating cost) over the next 12 days. Using your solution from part b what is $\mathbb{E}[X]$? (You may define a variable b to be the correct answer from part b).

6. PO-TA-TOES[16 points]

After destroying the One Ring, Samwise returned to the Shire and dedicated himself to his one true passion: growing potatoes. Like Sam, potatoes are very sturdy and can thrive in temperatures between 45°F and 95°F. However, the temperature in the Shire is a bit finicky. On any given day i, the temperature T_i, which remains the same throughout the day, follows a continuous uniform distribution between 40°F and 100°F, independent of any other days. Note that the expected value of T_i is $\mathbb{E}[T_i] = 70$ and its variance is $Var(T_i) = 300$.

(a) Sam is particularly worried about temperatures that are too cold, so he wants to upper bound the probability that on any given day, the temperature drops under 45°F. Use Markov's Inequality and the random variable $L_i = 100 - T_i$(the distance of the temperature from its maximum value) to help Sam derive a bound on the probability of the temperature being too cold.

$$\mathbb{P}\left(T_i \leq 45\right) \; \square_{\text{inequality}} \; \boxed{}_{\text{bound}}$$

(b) Sam loves potatoes too much to only look at low temperatures. Use Chebyshev's Inequality to derive a bound on the probability that on any given day the temperature is outside the interval 45°F and 95°F.

$$\mathbb{P}\left(T_i \notin (45,95)\right) \; \square_{\text{inequality}} \; \boxed{}_{\text{bound}}$$

(c) Upon further research, Sam discovers that potatoes can survive a maximum of 34(not-necessarily consecutive) days of exposure to non-ideal temperatures (outside the interval 45°F and 95°F) in the 100-day period required for potatoes to grow. That is, a 35th day of exposure would be fatal to his beloved spuds. Assume that Sam manages to improve temperature control such that $\mathbb{P}\left[\text{outside } (45°\text{F}, 95°\text{F})\right] = \frac{1}{5}$. Use Chernoff's Bound to upper bound the probability that Sam's harvest will **not** survive the 100-day period.

$\mathbb{P}\left(\textbf{not} \text{ survive}\right)$ [ ] inequality [ ] bound

(d) The "Bilbo Baggins Memorial Potato Competition" is in 1000 days, allowing Sam to grow exactly 10 batches of differently colored potatoes. Sam will win the competition if and only if all of his 10 batches survive. Use the Union Bound and your answer from part c to derive a lower bound on the probability that Sam will win the competition. Use c to refer to your answer from part c.

$\mathbb{P}\left(\text{win}\right)$ [ ] inequality [ ] bound

7. Grogu Snacking [10 points]

Grogu loves snacking and is always on the lookout for new foods to try. He finds his favorite snack—the Fish Lady's orange eggs—with probability θ_E.

Otherwise, the only other reasonable option is to 'hunt' for frogs, which Grogu does whenever he can't have eggs (i.e., with probability $1 - \theta_E$). Upon finding a frog, he may swallow it (with probability θ_S) or release it (with probability $1 - \theta_S$).

Mando is always observing Grogu in an attempt to ensure that Grogu is not eating the eggs. Unfortunately, he is rather unsuccessful. In just one day, he has observed Grogu eating eggs 3 times and hunting 6 times (5 frogs were swallowed, 1 was released). Grogu himself thought this was an impressive feat, considering his snacking habits are all independent of each other.

(a) Determine the likelihood function $\mathcal{L}$ for Mando's observations of Grogu's diet.
In writing the likelihood function, you may assume that the data described is given in the order it happened (i.e., Grogu ate eggs first, then swallowed frogs, then released frogs). [4 points]

(b) Which of the following best describes how to find the MLE in this problem? [2 points]

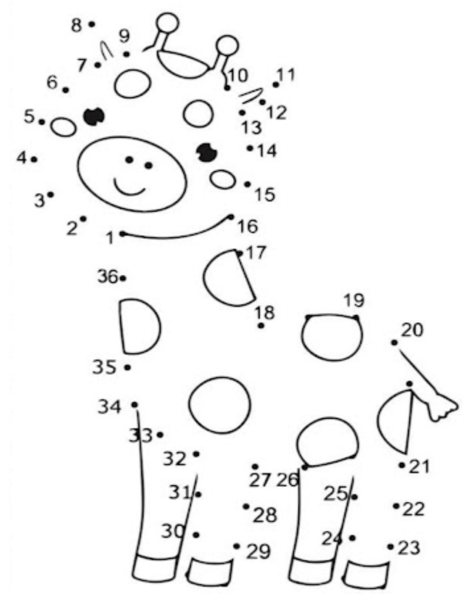
◯ Set the log-likelihood for $\theta_E = 0$ equal to the log-likelihood for $\theta_S = 0$ and solve.
◯ Set the likelihood equal to the log-likelihood and solve.
◯ Take the derivative of the likelihood and the log-likelihood, set each equal to 0 and solve the system of equations.
◯ Take partial derivatives of the log-likelihood with respect to θ_E and θ_S, set each equal to 0 and solve the system of equations.

(c) Suppose that θ_E is close to 1 (say $.99 < \theta_E < 1$). Which of these has the highest likelihood for five snacks? [2 points]
◯ All five eggs
◯ All five swallowed frogs
◯ All five released frogs
◯ No eggs, but we can't tell whether frogs would be swallowed or released

(d) Suppose that θ_E is close to $.5$ and θ_S is close to 1 (say $.99 < \theta_S < 1$). Which of these has the highest likelihood for ten snacks? [2 points]
◯ All ten eggs
◯ Five eggs and five swallowed frogs
◯ Five eggs and five released frogs
◯ Five swallowed frogs and five released frogs

8. Grading Morale [1 point]

Put something on this page, finish the image below, write a message to the teaching staff below, or do something else. Depending on your mood, this might be a poem, the statement of your favorite theorem, the statement of your least favorite theorem, or a piece of art. As long as you make some mark on this page, you will get the point. Looking at these helps keep the TAs happy while grading.



Have a relaxing spring break!

Use this page for extra space if needed. Be sure to tell us to look here on the original problem.