CSE 312 : Spring 2024 Midterm Exam, Form A

Name:	**NetID:** @uw.edu

Instructions

- You have ninety minutes to complete this exam.
- You are permitted one piece of 8.5x11 inch paper with handwritten notes (notes are allowed on both sides of the paper). In addition, a reference sheet is provided.
- You may not use a calculator or any other electronic devices during the exam.
- We will be scanning your exams before grading them. Please write legibly, and avoid writing up to the edge of the paper.
- If you run out of room, you may also use the last page for extra space, but tell us where to find your answer if it's not right below the problem.
- In general, show us the work you used to get to an answer; explanations will help us award partial credit, but we do not expect explanations at the level we usually require on homeworks.

Simplification Expectations

- Since you don't have a calculator for this exam, you do not have to do simplifications that could be done easily with a calculator. For example
$$\frac{\binom{5}{3} \cdot 17^2}{1-p} + 5^3$$
can be given as a final answer.
- However, answers which are much more complicated than the expected answer may receive deductions. For example: $\sum_{i=0}^{n} \binom{n}{i}$, $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}$ are **not** simplified sufficiently.

Advice

- Writing a few words about where an expression came from is often very helpful for awarding partial credit.
- Remember to take deep breaths.

Question	**Max points**
Short Answer	24
Counting	16
Probability	21
Conditioning	18
Random Variable	20
Grading Morale	1
Total	**100**

This page was intentionally left blank.

1. Multiple Choice and True/False

For the questions below,

- Questions with ◯ Circles have <u>exactly one</u> best answer. **Fully fill in** the circle for the <u>one</u> best answer.
- Questions with ☐ squares are "mark <u>all</u> that apply" questions. **Fully fill in** the square for <u>all</u> correct options.

(a) Your friend attempts to count the number of "two pair" hands. Two pair hands contain:

- Two cards of one value (e.g., two aces or two 8's)
- Two cards of a **different** value
- A fifth card of another different value.

For a standard 52 card deck (13 values, 4 suits), your friend says the number of two pair hands is

$$13\binom{4}{2} \cdot 12\binom{4}{2} \cdot 11\binom{4}{1}.$$

Which best describes their response?

◯ It overcounts—you need to divide by $5!$ for all possible reorderings.
◯ It overcounts—you need to divide by $2!$ for reordering the "first pair" compared to the "second pair"
◯ It undercounts—you need to multiply by $5!$ for all possible reorderings.
◯ It undercounts—you need to multiply by $2!$ for reordering the "first pair" compared to the "second pair"

(b) There are m houses in a suburban neighborhood. Suppose we need to pave a (direct) path between every possible pair of houses. How many paths need to be paved? (Once paved, a path can be used in both directions).

◯ m^2 paths
◯ $\frac{m(m-1)}{2}$ paths
◯ $\frac{m}{2}$ paths
◯ $m(m-1)$ paths
◯ None of the above.

(c) Suppose X is a random variable with support $\Omega_X = \{-1, 3\}$. Mark **all** of the options which **must** be true.

☐ The expectation of X is given by: $\mathbb{E}[X] = 2$.
☐ The support of X^2 is given by: $\Omega_{X^2} = \{1, 9\}$.
☐ The variance of X is non-negative, i.e., $\text{Var}(X) \geq 0$.

(d) Let A, B, and C be events. We say that A, B, C are "3-way independent" if

$$P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C)$$

Which of the following statements must be true (mark **ALL** that apply):

☐ If A, B, and C are 3-way independent, then they are pairwise independent.
☐ If A, B, and C are pairwise independent, then they are 3-way independent.
☐ If A, B, and C are pairwise and 3-way independent, then they are mutually independent.

- Questions with ◯ Circles have exactly one best answer. **Fully fill in** the circle for the one best answer.
- Questions with ☐ squares are "mark all that apply" questions. **Fully fill in** the square for all correct options.

(e) Suppose you assign 14 books to 4 bookshelves. Which of the following is true?

◯ There is **at least** 1 bookshelf with at most 3 books
◯ There are **at least** 4 bookshelves with **at least** 3 books
◯ There is at most 1 bookshelf with **at least** 4 books
◯ There is **at least** 1 bookshelf with **at least** 4 books
◯ There are **at least** 4 bookshelves with **at least** 1 book

(f) Let A, B be events with non-zero probability, which are independent. Which of the following is true?

◯ A and B are **always** mutually exclusive
◯ A and B are **sometimes** mutually exclusive
◯ A and B are **never** mutually exclusive

(g) Consider the following attempt at writing a CDF. Is it valid?

$$F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ \lfloor x \rfloor / 20 & \text{if } 0 \le x \le 10 \\ \frac{1}{2} + \lfloor (x - 10) \rfloor / 10 & \text{if } 10 < x \le 15 \end{cases}$$

◯ The CDF is valid as written.
◯ The CDF requires a "0 otherwise" case.
◯ The CDF requires a "1 otherwise" case.
◯ The CDF should not have the floor functions.

(h) Run the following experiment: flip a fair coin. If it is heads, roll a fair 6-sided die twice (independently); if it is tails, roll a fair 20-sided die twice (independently).
Let A be the event you roll a 10 on the first roll.
Let B be the event you roll a 10 on the second roll.
Let T be the event the coin comes up tails.

Mark **all** true statements below:

☐ A and B are independent.
☐ A and B are conditionally independent, conditioned on T.
☐ A and B are conditionally independent, conditioned on $\overline{T}$.

2. Party Time! [16 points]

Suppose you are throwing a party and you want to buy 30 balloons. The shop has five colors: red, blue, green, purple, orange. All balloons of a given color are identical (e.g., two purple balloons are indistinguishable from each other). You are allowed to take any number of any color as long as you hit the total of 30. Let g and b be the number of green and blue balloons you take respectively.

(a) How many balloon allocations are possible (with no additional restrictions)? [4 points]

(b) How many balloon allocations are possible such that **strictly less than** 3 of the balloons are green or blue (i.e. $g + b < 3$)? [5 points]
Hint: You might want to use casework where $g + b = 0, 1$ or 2.

(c) Let a and b be the **correct** answers for parts (a) and (b) respectively. How many balloon allocations are possible such that **at least** 3 of the ballons are green or blue (i.e. $g + b \geq 3$)? Your answer **must** be in terms of a and/or b. [2 points]

(d) 40 guests come to your party and enter a lucky draw. Winners are chosen such that a gold, silver, and bronze balloon are awarded as 1st, 2nd, and 3rd prizes respectively. 7 more guests are then awarded identical consolation white balloons (and the other 30 get nothing).
How many such prize assignments are possible? [5 points]

3. Squidward's Skittles [21 points]

Squidward has a bag of 50 **distinguishable** Skittles consisting of 10 of each of the 5 original flavors: `orange`, `lemon`, `apple`[1], `grape`, and `strawberry`.

Squidward wants to spice up his Skittle-eating experience. He draws 4 random Skittles from the bag to eat as a mix of flavors. However, Squidward detests tasting certain pairs of flavors together. In particular, Squidward will be *dissatisfied* if `{apple, grape}` or `{orange, apple}` flavor combinations are subsets of the flavors in the mix of 4 Skittles.

For example, if Squidward draws `{orange, lemon, grape, grape}` then he will be satisfied, but if he draws `{orange, orange, apple, apple}` then he will be dissatisfied.

For this problem, define events:

- A to be the event that Squidward's mix does not contain any `apple` Skittles
- G to be the event that Squidward's mix does not contain any `grape` Skittles

(a) Using the events defined above, what is the event that the mix contains at least one `apple` Skittle **and** at least one `grape` Skittle? Write your answer in the line below. [3 points]

____

(b) What is the **probability** that the mix contains at least one `apple` **and** at least one `grape` Skittle? [6 points]

Hint: Use the principle of inclusion-exclusion and the event you found in part (a). If your event in part (a) has an intersection, you may want to rewrite it as an equivalent expression using a union.

[1] green apple, but apple for short

This problem continues with the same setup from the last page. Here's the setup again:

Squidward has a bag of 50 **distinguishable** Skittles consisting of 10 of each of the 5 original flavors: `orange`, `lemon`, `apple`, `grape`, and `strawberry`.

Squidward wants to spice up his Skittle-eating experience. He draws 4 random Skittles from the bag to eat as a mix of flavors. However, Squidward detests tasting certain pairs of flavors. In particular, Squidward will be *dissatisfied* if `{apple, grape}` or `{orange, apple}` flavor combinations are subsets of the flavors in the mix of 4 Skittles.

For example, if Squidward draws `{orange, lemon, grape, grape}` then he will be satisfied, but if he draws `{orange, orange, apple, apple}` then he will be dissatisfied.

For this problem, define events:

- A to be the event that Squidward's mix does not contain any `apple` Skittles
- G to be the event that Squidward's mix does not contain any `grape` Skittles

Now back to problems

(c) What is the probability that Squidward gets at least one `apple`, at least one `grape`, and at least one `orange` Skittle? [6 points]

Hint: Consider two cases where either 1) all 4 Skittles are unique flavors or 2) there is one repeated flavor amongst the 4 Skittles in the mix.

(d) Let p_1 be the correct probability obtained in part (b) and p_2 be the correct probability obtained in part (c). In terms of p_1 and p_2, what is the probability that Squidward will be *satisfied* (i.e. **not** *dissatisfied*) with his Skittle-eating experience? [6 points]

4. Shrek Goes to Section [18 points]

Donkey is unfortunately falling behind in their assignments at this point of the quarter, and against his best judgment, he is strongly considering skipping section to catch up on studying. Donkey decides whether to go to section or not by asking Shrek, a friend in the same section.

- Shrek is a very reliable friend; he goes to section with probability s.
- If Shrek goes, Donkey is motivated and decides to go to section $8/9$ of the time.
- If Shrek doesn't go, Donkey is discouraged and skips section $4/7$ of the time.

Let D be the event that Donkey goes to section, and S be the event that Shrek goes to section.

(a) Which conditional probability or probability is referred to in the **last** bullet point?
Write the appropriate symbols below: [3 points]

$$\mathbb{P}\left(\underline{\hspace{3cm}}\right)$$

(b) What is the probability that Donkey goes to section?
Express your answer in terms of s. [5 points]

(c) If Donkey went to section, what is the probability that Shrek went to section?
Let d be the answer to part b. Express your answer in terms of d and s. [5 points]

(d) What is the probability that both Shrek and Donkey go to section?
You may express your answer in terms of previously defined variables. [5 points]

5. Random Variables [20 points]

A group of n friends is planning a book club. They have created a list of n books which they are considering reading. Each of the friends like exactly $n/2$ of the books (though which of the books could be different depending on which of the friends you ask). You may assume n is even so that $n/2$ is an integer. To form the club, each person will be given a set of 3 of the n books to read (each person gets three distinct books, but the same book can be assigned to any number of people). The set given to each person is uniformly random and independent of sets given to all others.

(a) Call a friend **very pleased** if they like **all** of the 3 books they read. What is the expected number of very pleased friends? [5 points]

(b) Once the books are assigned and read it's time to discuss! For each of the n books, every person who was assigned that book will meet to discuss. What is the expected number of books with **no** readers to discuss it? [5 points]

(c) Suppose that Edward is one of the friends. Define Y to be the number of books that Edward is assigned that he likes. Write the PMF for Y. Be sure to include all cases. [3 points]

This problem continues with the same setup from the last page. Here's the setup again:

A group of n friends is planning a book club. They have created a list of n books which they are considering reading. Each of the friends like exactly $n/2$ of the books (though which of the books could be different depending on which of the friends you ask). You may assume n is even so that $n/2$ is an integer. To form the club, each person will be given a set of 3 of the n books to read (each person gets three distinct books, but the same book can be assigned to any number of people). The set given to each person is uniformly random and independent of sets given to all others.

Now back to problems

(d) One of the books is *Probability and Statistics with Applications to Computing* (by Alex Tsun). Let X be the number of friends who are given that book.

(i) What is the support of X? [3 points]

(ii) what is Var (X)? [4 points]

6. Grading Morale [1 point]

Robbie is playing on the softball team this quarter! Draw a picture, write a poem, or make a meme of a 312 topic that is interesting to you that somehow incorporates softball.

Use this page for more room on any previous problem.
If you use this page, tell us to look here on the original problem!

Reference Sheet

Theorem: Binomial Theorem

Let $x, y \in \mathbb{R}$ and $n \in \mathbb{N}$ a positive integer. Then: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

Theorem: Principle of Inclusion-Exclusion (PIE)

2 events: $|A \cup B| = |A| + |B| - |A \cap B|$
3 events: $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$
k events: singles - doubles + triples - quads + ...

Theorem: Pigeonhole Principle

If there are n pigeons we want to put into k holes (where $n > k$), then at least one pigeonhole must contain at least 2 (or to be precise, $\lceil n/k \rceil$) pigeons.

Definition: Key Probability Definitions

The **sample space** is the set Ω of all possible outcomes of an experiment.
An **event** is any subset $E \subseteq \Omega$.
Events E and F are **mutually exclusive** if $E \cap F = \emptyset$.

Definition: Probability space

A *probablity space* is a pair $(\Omega, \mathbb{P})$, where Ω is the sample space
$\mathbb{P} : \Omega \to [0,1]$ is a *probability measure* such that $\sum_{x \in \Omega} \mathbb{P}(x) = 1$.
The probability of an event $E \subseteq \Omega$ is $\mathbb{P}(E) = \sum_{x \in E} \mathbb{P}(x)$.

Definition: Conditional Probability

$$\mathbb{P}[A \mid B] = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]}$$

Theorem: Bayes Theorem

$$\mathbb{P}[A \mid B] = \frac{\mathbb{P}[B \mid A]\,\mathbb{P}[A]}{\mathbb{P}[B]}$$

Definition: Partition

Non-empty events $E_1, \ldots, E_n$ **partition** the sample space Ω if:

- **(Exhaustive)** $E_1 \cup E_2 \cup \cdots \cup E_n = \bigcup_{i=1}^{n} E_i = \Omega$ (they cover the entire sample space).
- **(Pairwise Mutually Exclusive)** For all $i \neq j$, $E_i \cap E_j = \emptyset$ (none of them overlap)

Theorem: Law of Total Probability (LTP)

If events $E_1, \ldots, E_n$ partition Ω, then for any event F:

$$\mathbb{P}[F] = \sum_{i=1}^{n} \mathbb{P}[F \cap E_i] = \sum_{i=1}^{n} \mathbb{P}[F \mid E_i]\,\mathbb{P}[E_i]$$

Theorem: Bayes Theorem with LTP

Let events $E_1, \ldots, E_n$ partition the sample space Ω, and let F be another event. Then:

$$\mathbb{P}[E_1 \mid F] = \frac{\mathbb{P}[F \mid E_1]\,\mathbb{P}[E_1]}{\sum_{i=1}^{n} \mathbb{P}[F \mid E_i]\,\mathbb{P}[E_i]}$$

Definition: Independence (Events)

A and B are **independent** if any of the following equivalent statements hold:
1. $\mathbb{P}[A \cap B] = \mathbb{P}[A]\ \mathbb{P}[B]$
2. $\mathbb{P}[A \mid B] = \mathbb{P}[A]$
3. $\mathbb{P}[B \mid A] = \mathbb{P}[B]$

Theorem: Chain Rule

Let $A_1, \ldots, A_n$ be events with nonzero probabilities. Then:
$\mathbb{P}[A_1 \cap \cdots \cap A_n] =$
$\mathbb{P}[A_1]\,\mathbb{P}[A_2 \mid A_1]\,\mathbb{P}[A_3 \mid A_1 \cap A_2] \cdots \mathbb{P}[A_n \mid A_1 \cap \cdots \cap A_{n-1}]$

Definition: Mutual Independence (Events)

We say n events $A_1, A_2, \ldots, A_n$ are **(mutually) independent** if, for *any* subset $I \subseteq [n] = \{1, 2, \ldots, n\}$, we have

$$\mathbb{P}\left[\bigcap_{i \in I} A_i\right] = \prod_{i \in I} \mathbb{P}[A_i]$$

This equation is actually representing 2^n equations since there are 2^n subsets of $[n]$.

Definition: Conditional Independence

A and B are **conditionally independent given an event** C if any of the following equivalent statements hold:
1. $\mathbb{P}[A \cap B \mid C] = \mathbb{P}[A \mid C]\ \mathbb{P}[B \mid C]$
2. $\mathbb{P}[A \mid B \cap C] = \mathbb{P}[A \mid C]$
3. $\mathbb{P}[B \mid A \cap C] = \mathbb{P}[B \mid C]$

Definition: Random Variable (RV)

A random variable (RV) X is a numeric function of the outcome $X : \Omega \to \mathbb{R}$. The set of possible values X can take on is its **range/support**, denoted Ω_X.

Definition: Probability Mass Function (PMF)

For a discrete RV X, assigns probabilities to values in its range. That is $p_X : \Omega_X \to [0,1]$ where: $p_X(k) = \mathbb{P}[X = k]$.

Definition: Expectation

The **expectation** of a discrete RV X is: $\mathbb{E}[X] = \sum_{k \in \Omega_X} k \cdot p_X(k)$.

Theorem: Linearity of Expectation (LoE)

For any random variables X, Y (possibly dependent):
$$\mathbb{E}[aX + bY + c] = a\mathbb{E}[X] + b\mathbb{E}[Y] + c$$

Theorem: Law of the Unconscious Statistician (LOTUS)

For a discrete RV X and function g, $\mathbb{E}[g(X)] = \sum_{b \in \Omega_X} g(b) \cdot p_X(b)$.

Definition: Variance

$\text{Var}(X) = \mathbb{E}\left[(X - \mathbb{E}[X])^2\right] = \mathbb{E}\left[X^2\right] - \mathbb{E}[X]^2$.

Theorem: Property of Variance

$\text{Var}(aX + b) = a^2 \text{Var}(X)$.

Definition: Independence (Random Variables)

Random variables X and Y are **independent** if for *all* $x \in \Omega_X$ and all $y \in \Omega_Y$:
$$\mathbb{P}[X = x, Y = y] = \mathbb{P}[X = x] \cdot \mathbb{P}[Y = y].$$

Theorem: Variance Adds for Independent RVs

If X, Y are independent, then $\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y)$.