

CSE 312 : Spring 2023 Midterm Exam Solutions

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Instructions

- You have ninety minutes to complete this exam.
- You are permitted one piece of 8.5x11 inch paper with handwritten notes (notes are allowed on both sides of the paper).
- You may not use a calculator or any other electronic devices during the exam.
- We will be scanning your exams before grading them. Please write legibly, and avoid writing up to the edge of the paper.
- If you run out of room, you may also use the last page for extra space, but tell us where to find your answer if it's not right below the problem.
- Since you don't have a calculator, you are generally free to **not** simplify expressions (though you may if you think it will be helpful).
- In general, show us the work you used to get to an answer, and explanations will help us reward partial credit, but we do not expect explanations at the level we usually require on homeworks.

Advice

- Writing a few words about where an expression came from is often very helpful for awarding partial credit.
- Remember to take deep breaths.

Question	Max points
Short Answer	24
Counting	15
Probability	20
Conditioning	20
Random Variable	25
Grading Morale	1
Total	105

1. [Multiple Choice] A Variance of Topics [24 points]

(a) Suppose $k > 0$ pigeons each go into one of $n > 0$ pigeonholes, numbered 1 through n . (Each pigeonhole can have zero, one, or multiple pigeons in it.) Then...

- Pigeonhole 1 will always have exactly $\lceil \frac{k}{n} \rceil$ pigeons in it.
- Pigeonhole 1 will always have **at least** $\lceil \frac{k}{n} \rceil$ pigeons in it (but may not be exactly that many).
- Neither statement is true.

Solution:

Neither statement is true.

The pigeonhole principle states that *some* pigeonhole will have **at least** $\lceil \frac{k}{n} \rceil$ pigeons in it, but it need not be pigeonhole 1.

(b) Suppose events A and B are independent (with $0 < \mathbb{P}(A) < 1$ and $0 < \mathbb{P}(B) < 1$). Then, events A and \bar{B} are independent.

- Always true
- Sometimes true
- Never true

Solution:

Always true.

$$\begin{aligned}
 \mathbb{P}(A)\mathbb{P}(B^C) &= \mathbb{P}(A)(1 - \mathbb{P}(B)) && \text{Complementation} \\
 &= \mathbb{P}(A) - \mathbb{P}(A)\mathbb{P}(B) \\
 &= \mathbb{P}(A) - \mathbb{P}(A \cap B) && A \text{ and } B \text{ are independent} \\
 &= \mathbb{P}(A \cap B^C)
 \end{aligned}$$

(c) Consider the following potential PMF for a random variable X :

$$p_X(k) = \begin{cases} \frac{1}{4} & k = 3, 6 \\ \frac{3}{4} & k = 4 \\ 0 & \text{otherwise} \end{cases}$$

- It is not a valid PMF because it needs floors in the ranges for k .
- It is not a valid PMF because PMFs must sum to 1.
- It is not a valid PMF because $\lim_{k \rightarrow \infty} p_X(k) \neq 1$.
- It is a valid PMF. **Solution:**

It is not a valid PMF because PMFs must sum to 1.

The sum of the probabilities over the range of X does not add to 1. $\mathbb{P}(X = 3) + \mathbb{P}(X = 4) + \mathbb{P}(X = 6) = \frac{1}{4} + \frac{3}{4} + \frac{1}{4} = \frac{5}{4} > 1$.

(d) The probability of picking a card that is a queen or a diamond from a standard deck of 52 cards is

$$\frac{4}{52} + \frac{13}{52}$$

- True, this is an application of the sum rule.
 True, this is an application of inclusion-exclusion.
 True, this is an application of complementary counting
 False **Solution:**

False.

Cards that are queens and cards that are diamonds are not disjoint sets, so we must use inclusion-exclusion.

(e) True or false:

$$\sum_{n=0}^{75} (-2)^{75-n} \binom{75}{n} = 1$$

- True
 False

Solution:

False.

By binomial theorem this is equal to $(-2 + 1)^{75} = -1$.

(f) True or false: the number of ways a people can buy one donut each with b flavor options is a^b .

- True
 False

Solution:

False.

Each person has b options, and there are a people, so there are b^a possible orders.

(g) If X is an indicator random variable, then $\mathbb{E}[X]$ is

- Less than $\mathbb{E}[X^2]$
 Greater than $\mathbb{E}[X^2]$
 Equal to $\mathbb{E}[X^2]$
 Answer depends on the particular indicator random variable.

Solution:

equal to $\mathbb{E}[X^2]$

Since X is an indicator RV, $\Omega_X = \{0, 1\}$. Then $\mathbb{E}[X] = 1 \cdot \mathbb{P}(X = 1) + 0 \cdot \mathbb{P}(X = 0) = \mathbb{P}(X = 1)$ by definition of expectation. Similarly, $\mathbb{E}[X^2] = 1^2 \cdot \mathbb{P}(X = 1) + 0^2 \cdot \mathbb{P}(X = 0) = \mathbb{P}(X = 1)$, by LOTUS / definition of expectation.

(h) Suppose you independently roll a fair red die and a fair blue die (both with 6 sides). Let E be the event that the red die shows a 3, F be the event the blue die shows a 5, and G be the event the sum is 8. Which of the following is true?

- $\mathbb{E}[G] = \mathbb{E}[E] + \mathbb{E}[F]$.
- F and G are dependent.
- E and F are dependent.
- None of the above.

Solution:

F and G are dependent.

$\mathbb{P}(F) = \frac{1}{6}$. $\mathbb{P}(F|G) = \frac{1}{5}$. Since $\mathbb{P}(F|G) \neq \mathbb{P}(F)$, F and G are not independent.

2. [Counting] I Love Bubble Tea [15 points]

- (a) Marlene wants to buy 20 bubble teas for a party. There are 4 flavors available - brown sugar, matcha, lychee, and strawberry. If she wants to buy at least 2 of each flavor, and if bubble teas of the same flavor are **indistinguishable**, how many ways are there to select the 20 drinks? **Solution:**

We first select two drinks of each flavor - since the drinks of the same flavor are indistinguishable, there is exactly one way to do this. Then, we select the remaining $20 - (4 \cdot 2) = 12$ drinks. Since the drinks of the same flavor are indistinguishable, we can use the stars and bars approach, with 12 stars and 4 bins. This gives us $\binom{12+(4-1)}{4-1}$ options.

- (b) Now, Marlene goes to a different bubble tea shop where each drink is a combination of 4 distinct flavors. In this shop, there are 10 different flavors (two of which are brown sugar and matcha). How many ways can Marlene order 1 drink, such that the drink she ordered contains brown sugar flavor, and/or contains matcha flavor? Consider two orders the same they have the same combination of flavors. **Solution:**

Let B be the set of orders where each drink includes the brown sugar flavor, and M be the set of orders where each drink includes the matcha flavor. $|B| = \binom{9}{3}$ because we need 3 additional flavors from the remaining 9. Similarly, $|M| = \binom{9}{3}$. $|B \cap M| = \binom{8}{2}$ because there is already both matcha and brown sugar, so we need 2 additional flavors from the remaining 8. By inclusion-exclusion $|M \cup B| = |M| + |B| - |M \cap B| = \binom{9}{3} + \binom{9}{3} - \binom{8}{2}$.

Alternatively, we can do complementary counting to find all ways to pick 4 flavors, and then subtract the ways that doesn't have either. This would give $\binom{10}{4} - \binom{8}{4}$

3. [Probability] Society of Cards [20 points]

Suppose we have a deck of cards consisting of the 4 suits ($\clubsuit, \diamond, \heartsuit, \spadesuit$), each with cards ranked from 1 to 13, plus 2 distinguishable Joker cards (Joker A, Joker B). The deck has $(13 \cdot 4) + 2 = 54$ cards total. Suppose you draw a hand of 5 cards from this deck, with each hand equally likely to be drawn.

- (a) What is the probability of your hand containing **at least 1 Joker**? **Solution:**

$$\begin{aligned} \mathbb{P}(\text{at least 1 Joker}) &= 1 - \mathbb{P}(\text{no Jokers}) \\ &= 1 - \frac{\# \text{ of possible hands w/ no Jokers}}{\# \text{ of possible hands total}} \\ &= 1 - \frac{\binom{52}{5}}{\binom{54}{5}} \end{aligned}$$

Another solution:

$$\begin{aligned} \mathbb{P}(\text{at least 1 Joker}) &= \mathbb{P}(\text{exactly 1 Joker}) + \mathbb{P}(\text{exactly 2 Jokers}) \\ &= \frac{\# \text{ of possible hands w/ exactly 1 Joker}}{\# \text{ of possible hands total}} + \frac{\# \text{ of possible hands w/ 2 Jokers}}{\# \text{ of possible hands total}} \\ &= \frac{\binom{2}{1} \binom{52}{4}}{\binom{54}{5}} + \frac{\binom{2}{2} \binom{52}{3}}{\binom{54}{5}} \end{aligned}$$

Another solution:

$$\begin{aligned} \mathbb{P}(\text{at least 1 Joker}) &= 1 - \mathbb{P}(\text{no Jokers}) \\ &= 1 - \mathbb{P}(\text{Card 1 not Joker}) \mathbb{P}(\text{Card 2 not Joker} \mid \text{Card 1 not Joker}) \\ &\quad \cdots \mathbb{P}(\text{Card 5 not Joker} \mid \text{Cards 1-4 not Jokers}) \\ &= 1 - \frac{52}{54} \cdot \frac{51}{53} \cdot \frac{50}{52} \cdot \frac{49}{51} \cdot \frac{48}{50} \end{aligned}$$

- (b) Let J be the event that your hand contains **exactly 1 Joker**. Let S be the event that your hand contains the “1 of Spades” (“1♠”) card. Find $\mathbb{P}(J \cup S)$. **Solution:**

$$\begin{aligned} \mathbb{P}(J \cup S) &= \mathbb{P}(J) + \mathbb{P}(S) - \mathbb{P}(J \cap S) \\ &= \frac{\# \text{ hands w/ exactly 1 Joker}}{\# \text{ possible hands}} + \frac{\# \text{ hands with "1♠"}}{\# \text{ possible hands}} - \frac{\# \text{ hands with both "1♠" and exactly 1 Joker}}{\# \text{ possible hands}} \\ &= \frac{\binom{2}{1} \binom{52}{4}}{\binom{54}{5}} + \frac{\binom{1}{1} \binom{53}{4}}{\binom{54}{5}} - \frac{\binom{1}{1} \binom{2}{1} \binom{51}{3}}{\binom{54}{5}} \end{aligned}$$

- (c) What is the probability that your hand is a “Straight”, that is, a hand where the cards’ ranks form a consecutive sequence?

For example, some **valid** Straights are: $\{8\heartsuit, 9\clubsuit, 10\clubsuit, 11\diamond, 12\spadesuit\}$ and $\{1\heartsuit, 2\heartsuit, 3\heartsuit, 5\heartsuit, 4\heartsuit\}$.

Some hands that are **not valid** Straights are: $\{10\clubsuit, 11\spadesuit, 12\heartsuit, 13\spadesuit, 1\heartsuit\}$, $\{4\clubsuit, 5\diamond, 6\spadesuit, 7\heartsuit, 9\clubsuit\}$, $\{6\diamond, 7\diamond, 8\clubsuit, 8\diamond, 9\diamond\}$, $\{2\spadesuit, 4\heartsuit, 6\spadesuit, 8\spadesuit, 10\heartsuit\}$, and $\{10\clubsuit, 11\spadesuit, 12\heartsuit, 13\spadesuit, \text{Joker A}\}$.

Recall that card order in a hand doesn’t matter. A straight hand cannot have any Jokers.

Solution:

There are 9 choices for the rank of the lowest card in a Straight hand (can't form a Straight with 10, 11, 12, or 13 as our lowest card). Given this choice, there's only 1 possibility for the ranks of the other cards. There are 4^5 ways to choose the suits for the 5 cards (since they all have different ranks). Using product rule to combine these steps,

$$\mathbb{P}(\text{Straight}) = \frac{|E|}{|\Omega|} = \frac{9 \cdot 1 \cdot 4^5}{\binom{54}{5}}$$

4. [Conditioning] Superstition or Reality? [20 points]

Allie and her daughter Sophia love watching Kraken ice hockey games. For some games, one of them will wear the lucky tentacle¹, which affects the Kraken's chances of winning.

- If neither wears the lucky tentacle, the Kraken win with probability $1/2$.
- If Allie wears the lucky tentacle, the Kraken win with probability p (you'll solve for the constant p later).
- If Sophia wears the lucky tentacle, the Kraken win with probability $2p$.
- Note that these are the only possibilities (there is no way for both to wear the one lucky tentacle).

Let A, S, N be the events that Allie, Sophia, or neither wear the lucky tentacle (respectively). Let W be the event that the Kraken win.

- (a) Which conditional probability or probability is referred to in the first bullet? Write the appropriate symbols below:

$$\mathbb{P}(\underline{\hspace{2cm}})$$

Solution:

$$\mathbb{P}(W|N)$$

- (b) Sophia has an early bed-time, so she only wears the tentacle $1/5$ of the time. Allie wears it $2/5$ of the time and the remaining $2/5$ neither do. We also know that the Kraken won $\frac{13}{25}$ of their games. Write an expression that will let you solve for p , in terms of only p and numerical values. **Solution:**

$$\mathbb{P}(W) = \mathbb{P}(W|A)\mathbb{P}(A) + \mathbb{P}(W|S)\mathbb{P}(S) + \mathbb{P}(W|N)\mathbb{P}(N)$$

$$\frac{13}{25} = p \cdot \frac{2}{5} + 2p \cdot \frac{1}{5} + \frac{1}{2} \cdot \frac{2}{5}$$

By Law of Total Probability partitioned on A, S and N.

- (c) Now, solve that formula for p . **Solution:**

$$\frac{13}{25} = p \cdot \frac{2}{5} + 2p \cdot \frac{1}{5} + \frac{1}{2} \cdot \frac{2}{5}$$

$$\frac{13}{25} = \frac{4p + 1}{5}$$

$$13 = (4p + 1) \cdot 5$$

$$13 = 20p + 5$$

$$p = \frac{8}{20} = \frac{2}{5}$$

- (d) If the Kraken won, what is the probability that Sophia was wearing the lucky tentacle?

Write a formula to represent this probability **using only notation** (no numbers nor expressions using p ; just probabilities, conditional probabilities, and events) that describe the situation. **Solution:**

¹The tentacle is the Kraken's replacement for a foam finger. A foam finger is a giant hand-shaped piece of foam to say your team is "#1". You don't need to understand this for the problem

$$\mathbb{P}(S|W) = \frac{\mathbb{P}(W|S) \cdot \mathbb{P}(S)}{\mathbb{P}(W)}$$

- (e) Now plug in numbers/expressions in terms of p for each of the parts in the formula you wrote in part (d). **You do not need to simplify**, but may if you think it would help you check your work. **Solution:**

$$\mathbb{P}(S|W) = \frac{\mathbb{P}(W|S) \cdot \mathbb{P}(S)}{\mathbb{P}(W)}$$

$$\mathbb{P}(S|W) = \frac{2p \cdot \frac{1}{5}}{\frac{13}{25}} \text{ or } \mathbb{P}(S|W) = \frac{\frac{4}{5} \cdot \frac{1}{5}}{\frac{13}{25}}$$

5. [Random Variables] Robbie Can't Sing [25 points]

A group of 5 friends visits a karaoke lounge. The karaoke machine has 20 songs. The friends share the same opinion on every song, so of the available songs: 12 are songs they like, while the remaining 8 are ones they dislike. The group sets the playlist on shuffle, such that every ordering of the 20 songs is equally likely. They decide to take turns singing 4 (consecutive) songs each. A person is **unhappy** if they dislike all 4 songs they sing.

- (a) What is the probability that the first person to sing is **unhappy**? **Solution:**

$$\Pr(U_1) = \frac{\binom{8}{4}}{\binom{20}{4}}$$

- (b) What is the expected number of **unhappy** friends once everyone is done singing?

Solution:

Let U be the number of unhappy friends. Let U_i be an indicator that is 1 if the i th friend is unhappy, and 0 otherwise.

$$\mathbb{E}[U] = \sum_{i=1}^5 \mathbb{E}[U_i] = 5 \cdot \frac{\binom{8}{4}}{\binom{20}{4}}$$

Now, suppose the 20th song is followed by the 1st song. The shuffle order is the same when looping back to the first song that was played. This means that every song i (where $1 \leq i \leq 20$) is always immediately preceded by one song and immediately followed by another song.

Let X_k be the number of liked songs played immediately before or immediately after the k th disliked song (recall that each song has exactly one “immediately before” and “immediately following” it).

- (c) What is Ω_{X_k} , i.e. the support of X_k ?

Solution:

$$\Omega_{X_k} = \{0, 1, 2\}$$

- (d) Find the PMF of X_k . **Solution:**

$$p_{X_k}(x) = \begin{cases} \frac{7}{19} \cdot \frac{6}{18} & x = 0 \\ \binom{2}{1} \cdot \frac{12}{19} \cdot \frac{7}{18} & x = 1 \\ \frac{12}{19} \cdot \frac{11}{18} & x = 2 \end{cases}$$

- (e) Find the expected number of liked songs played immediately before or immediately after a disliked song using your answer to the previous question. **Solution:**

$$\mathbb{E}[X_k] = 1 \cdot \binom{2}{1} \cdot \frac{12}{19} \cdot \frac{7}{18} + 2 \cdot \frac{12}{19} \cdot \frac{11}{18} = \frac{24}{19}$$

- (f) A *switch* occurs when a liked song is immediately next to a disliked song. For example, consider the following sequences:

- Disliked Disliked Liked - this sequence contains 1 switch
- Liked Disliked Liked - this sequence contains 2 switches

What is the expected number of times we switch between disliked and liked songs if we play 21 songs (the 20 available songs, followed by the one we started with)? You may use the variable e to represent the answer to part (e) in your answer. **Solution:**

Let Y be the total number of times we switch between disliked and liked songs. We switch once for every liked song immediately before and after a disliked song. Then, $Y = \sum_{k=1}^8 X_k$. Using linearity of expectation,

$$\mathbb{E}[Y] = \mathbb{E}\left[\sum_{i=1}^8 X_k\right] = \sum_{i=1}^8 \mathbb{E}[X_k] = 8 \cdot \frac{24}{19} = 8e$$