Homework 8: MLEs and Joint Distributions

For each problem, remember you must briefly explain/justify how you obtained your answer, as correct answers without an explanation will not receive full credit. Moreover, in the event of an incorrect answer, we can still try to give you partial credit based on the explanation you provide.

In general, your goal in an explanation is to write enough that a student from class who has attended lecture, but not read the problem yet, could understand your approach, verify your reasoning, and believe your answer is correct. While we do not usually need to see arithmetic, you must include enough work that in principle one could rederive your answer with only a scientific calculator. For each problem, make sure to explicitly define all random variables you use, and be clear about how they are related to each other using proper notation (conditionals, summations, etc.).

Unless a problem states otherwise, you should leave your answer in terms of factorials, combinations, etc., for instance 26^7 or $26!/7!$ or $26 \cdot \binom{26}{7}$ are all good forms for final answers.

Instructions as to how to upload your solutions to gradescope are on the course web page.

Remember that you must tag your written problems on Gradescope.

Submission: You must upload a **pdf** of your written solutions to Gradescope under "HW 8". (Instructions as to how to upload your solutions to gradescope are on the course web page.) The use of latex is *highly recommended*. (Note that if you want to hand-write your solutions, you'll need to scan them. We will take off points for hand-written solutions that are difficult to read due to poor handwriting and neatness.)

Due Date: This assignment is due Wednesday December 3 at 11:59 PM.
Late Deadline: The last day to submit will be Saturday December 6. You will submit the written problems as a PDF to gradescope. Please put each numbered problem on its own page of the pdf (this will make selecting pages easier when you submit), and ensure that your pdfs are oriented correctly (e.g. not upside-down or sideways).

0. Extra Instructions :o

For calculations that require evaluating integrals (unless we indicate otherwise), you must

(a) Show the integral to evaluate (e.g., $\int_0^2 z \cdot 2dz$)

(b) Show an antiderivative and the values to evaluate at (e.g., $z^2|_0^2$)

(c) Plug in the values and simplify (e.g., $2^2 - 0^2 = 4$)

This is not a problem, so nothing needs to be submitted here.

1. Gradescope Questions

Answer the questions on gradescope about covariance and conditional expectations.

2. Maximally Liked Fetch Player [15 points]

Three TAs – Emma, Michael, and Jai – are playing fetch with the world famous husky Leia. They will throw a ball, and Leia will bring it back to one of the three at random for the honor of throwing the ball next.

Suppose that each round Leia independently chooses to have Emma throw with probability θ_E, Michael throw with probability θ_M, and Jai with probability $1-\theta_E-\theta_M$. (Thus, $0 \leq \theta_E+\theta_M \leq 1$). The parameters θ_E, θ_M are unknown. Suppose that $y_1, ..., y_n$ are n independent, identically distributed samples from this distribution of TA choices. Let x_E be the number of y_i's equal to Emma, let x_M be the number of y_i's equal to Michael, and let x_J be the number of y_i's equal to Jai. What are the maximum likelihood estimates for θ_E and θ_M in terms of x_E, x_M, and x_J?

In doing this problem, you do not need to do a second-derivative test (or any other test) to confirm you have a maximizer. You may assume any critical point you find is a maximizer.

For this problem, except where noted, all of your work must be understandable without reference to any calculator (including Wolfram-Alpha). You may check your answers using calculators, but your explanation may not rely on them.

(a) Write the Likelihood function. [3 points]

(b) Write the log-likelihood. [2 points]

(c) Give two equations describing the maximizer of the log-likelihood. Be sure to show work (e.g., taking of partial derivatives) on how you got these equations. [8 points]

(d) You should now have a system of two (linear) equations in two unknowns. Find the solution to this system, and use it to write the MLEs for θ_E and θ_M. You do **not** have to show your work for this part, and may use online algebra solvers if you wish. [2 points]

3. Unbiased Estimators [15 points]

Let $X_1, X_2, ..., X_n$ be random samples drawn from the continuous distribution Uniform$(0, \theta)$, for $\theta > 0$.

(a) For outcomes $x_1, \ldots, x_n$, compute the likelihood $\mathcal{L}(x_1, \ldots, x_n; \theta)$, making sure to include all cases. From there, deduce the MLE estimator $\hat{\theta}$ for θ.
Hint: the maximizer of a function can occur at endpoints, points of non-differentiability, or critical points.

(b) Is your MLE estimator from the previous part unbiased?
Hint: You may find section 7, problem 11 useful.

(c) Finally, consider the estimator

$$\tilde{\theta} = \left(2 \cdot \sum_{i=1}^{n} X_i/n\right)$$

Is $\tilde{\theta}$ an unbiased estimator for θ?

4. Conditional Expectations [15 points]

Let X and Y be continuous random variables with joint PDF:

$$f_{X,Y}(x, y) = \begin{cases} \frac{3}{4}x^2 & 0 \leq x \leq y \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

(a) What is $f_{Y|X}(y|x)$?

(b) Determine $\mathbb{E}[Y|X = x]$.

(c) What is $\mathbb{E}[Y]$?

5. Joint Continuous Densities [12 points]

Let X and Y be continuous random variables with the following joint distribution.

$$f_{X,Y}(x,y) = \begin{cases} \frac{2}{81}xy^3 & 0 \le x \le 2,\ 0 \le y \le 3 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find $\mathbb{P}(X > Y)$. [5 points]

(b) What is the marginal PDF of X? [3 points]

(c) What is the marginal PDF of Y? [3 points]

(d) Are X and Y independent? [1 point]

6. Duck Hunt [12 points]

12 hunters are waiting for ducks to fly by. When a flock of ducks flies overhead and hunters fire at the same time, but each chooses his target at random, independently of the others. Each hunter independently hits their target with probability 0.6.

(a) Assume that there are exactly d ducks flying by. What is the expected number of ducks that are hit? [6 points]

(b) Now, assume that the number of ducks in a flock is a Poisson random variable D with mean 7. Use the law of total expectation to compute the expected number of ducks that are hit. You may leave your answer as a summation. [6 points]

7. A Dependent Sum [5 points]

Suppose I repeatedly roll a fair 6-sided die. Let $X_1, X_2, \ldots$ be the outcomes of the rolls, and let N be the number of rolls I make **before** rolling a 6 for the first time. That is, if I see 6 on my first roll then $N = 0$. Compute

$$\mathbb{E}\left[\sum_{i=1}^{N} X_i\right].$$

8. Feedback [1 point]

Answer these questions on the separate Gradescope box for this question.

Please keep track of how much time you spend on this homework and answer the following questions. This can help us calibrate future assignments and future iterations of the course, and can help you identify which areas are most challenging for you.

- How many hours did you spend working on this assignment (excluding any extra credit questions, if applicable)? Report your estimate to the nearest hour.
- Which problem did you spend the most time on?
- Any other feedback for us?