Homework 2: More Counting and Probability

For each problem, remember you must briefly explain/justify how you obtained your answer, as correct answers without an explanation will not receive full credit. Moreover, in the event of an incorrect answer, we can still try to give you partial credit based on the explanation you provide.

In general, your goal in an explanation is to write enough that a student from class who has attended lecture, but not read the problem yet, could understand your approach, verify your reasoning, and believe your answer is correct. While we do not usually need to see arithmetic, you must include enough work that in principle one could rederive your answer with only a scientific calculator.

Unless a problem states otherwise, you should leave your answer in terms of factorials, combinations, etc., for instance 26^7 or $26!/7!$ or $26 \cdot \binom{26}{7}$ are all good forms for final answers.

Submission: You must upload a **pdf** of your written solutions to Gradescope under "HW 2 [Written]". (Instructions as to how to upload your solutions to Gradescope are on the course web page.) The use of LaTeX is *highly recommended*. (Note that if you want to hand-write your solutions, you'll need to scan them. We will take off points for hand-written solutions that are difficult to read due to poor handwriting and neatness.)

You will submit the written problems as a PDF to Gradescope. Please put each numbered problem on its own page of the pdf (this will make selecting pages easier when you submit), and ensure that your pdfs are oriented correctly (e.g. not upside-down or sideways).

Due Date: This assignment is due Wednesday October 8 at 11:59 PM.

Collaboration: Please read the full collaboration policy. If you work with others (and you should!), you must still write up your solution independently and name all of your collaborators somewhere on your assignment.

1. Combinatorial Identities [16 points]

Prove each of the following identities using a *combinatorial argument* (i.e., an argument that counts two different ways); an algebraic solution will be marked substantially incorrect.

For the purposes of these problems, using commutativity of multiplication and addition (i.e. $ab = ba$, $a + b = b + a$), and distributivity/factoring ($a(b + c) = ab + ac$) are allowed as part of a combinatorial argument. Any other algebra facts (e.g. Pascal's Rule about combinations, the definition of combinations/permutations in terms of factorials, canceling numbers that appear in numerators and denominators) would make it an algebraic solution, not a combinatorial one.

(a) $\displaystyle\sum_{i=0}^{k} \binom{k}{i}\binom{m}{i} = \binom{k+m}{k}$. You may assume that $k \geq m \geq 0$.
Hint: Start with the right-hand side and imagine you are choosing k club's memebers for this new academic year, from a group consisting of the k members from last year and m new applicants.

(b) $\displaystyle\sum_{i=x}^{y} \binom{y}{i}\binom{i}{x} = \binom{y}{x}2^{y-x}$. Assume that $y \geq x \geq 0$.
Hint: Think about choosing a group of winter TAs of varying size from y applicants with a fixed number of lead TAs.

(c) $\displaystyle\sum_{i=3}^{n-3} \binom{n-i}{3}\binom{i-1}{2} = \binom{n}{6}$. Assume that $n \geq 6$.

Hint: Think about choosing 6 numbers from a set of n numbers.

2. Chess Playing Pigeons

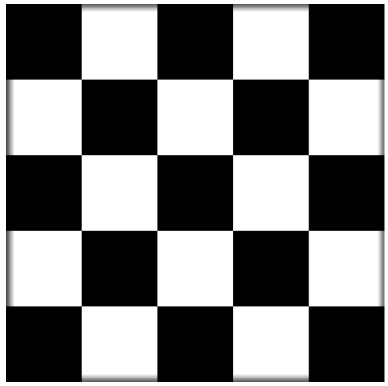

Figure 1: $n \times n$ Chess Board for $n = 5$

Suppose we have an $n \times n$ chess board and there is a pigeon on each square (so n^2 total pigeons). When the clock strikes midnight, each pigeon moves randomly to a valid adjacent square horizontally or vertically (but not diagonally).

(a) Suppose $n = 2k$ is an even integer. Show that it is possible for each square to still have exactly one pigeon after midnight.

(b) Suppose now that $n = 2k + 1$ is an odd integer. Show that it is guaranteed that two or more pigeons end up on the same square after midnight.

Hint: Think about the colors of the chess board. It's interesting that the birds we chose are pigeons.

3. Stuff into stuff [12 points]

(a) We have 22 (distinguishable) frogs and 48 (distinguishable) logs. How many different ways are there to assign the (distinguishable) frogs to the (distinguishable) logs? (Any number of frogs can go onto any of the logs.)

(b) We have 40 identical (indistinguishable) cookies. How many different ways are there to place the cookies into 13 (distinguishable) cookie jars? (Any number of cookies can go into any of the jars.)

(c) We have 58 identical (indistinguishable) cookies. How many different ways are there to place the cookies into 11 (distinguishable) cookie jars, if each jar is required to have at least four cookies in it?

4. Miscounting [14 points]

Consider the question: How many **7-card** poker hands (order doesn't matter) are there that contain at least two 3-of-a-kinds (3-of-a-kind means three cards of the same value). For example, this would be a valid hand: ace of hearts, ace of diamonds, ace of spaces, 7 of clubs, 7 of spades, 7 of hearts, and queen of clubs. (Note that a hand consisting of all 4 aces and three of the 7s is also valid.)

Here is how we might compute this:

> To compute the number of hands, apply the product rule using the following sequential process. First pick two ranks that have a 3-of-a-kind (e.g. ace and 7 in the example above). For the lower rank of these, pick the suits of the three cards. Then for the higher rank of these, pick the suits of the three

cards. Then out of the remaining $52 - 6 = 46$ cards, pick one. Therefore there are

$$\binom{13}{2} \cdot \binom{4}{3} \cdot \binom{4}{3} \cdot \binom{46}{1}$$

hands.

In this problem, you will find what is wrong with this solution.

(a) Is there overcounting in the solution? That is, is there a hand that can be produced by multiple outcomes of the sequential process? If there is, give one concrete example of such a hand and two outcomes of the process that produce it. If there is not, briefly (1-2 sentences) explain why there isn't.

(b) Is there undercounting in the problem? That is, is there a hand that cannot be produced by any outcomes of the sequential process? If there is, give one concrete example of such a hand and briefly explain why no outcome produces it. If there is no such hand, briefly explain why all hands are produced at least once.

(c) Correct the calculation – in this part you should produce a correct overall formula by subtracting/dividing out any errors that would fit in (a) and adding/multiplying in any errors that would fit in (b).

(d) Find the answer differently – take a different approach to counting this problem (e.g. use a different sequential process). Verify that you get the same number (via a different formula) than the last part.

5. Sample Spaces and Probabilities [online]

Do the gradescope question asking for sizes of sample spaces and probability computations. We will not be able to award partial credit, so please enter your submissions carefully.

6. Binomial Theorem applications [15 points]

For part (a) of this question (as with many others for 312), you could find the numerical answer in a few seconds by asking wolfram alpha (in this case, by asking it to expand the polynomial). We have learning goals associated with this problem that mean we want you to practice solving this problem by hand even though you could easily answer it with computational power.

Remember that you must give an explanation of an answer such that another student would understand the **principles** that go into solving the problem, and such that they could find the answer with a simple calculator (that doesn't have an "expand a polynomial" operation)

You may find it beneficial to verify your answer using WolframAlpha, but you may not use the "show steps" option on WolframAlpha or any similar tool.

(a) What is the coefficient of x^5y^{10} in the expansion of $(2x - y^2)^{10}$?

(b) Use the binomial theorem to prove that

$$\sum_{i=0}^{100} \binom{100}{i} (-4)^{100-i} = 3^{100}$$

7. Real-World: Is this a pigeon? [6 points]

The pigeonhole principle is surprisingly powerful and can be used to prove a variety of things - find one application of the principle that interests you (it can be CS-related or not, serious or not; the only requirements are that (1) you must find it interesting and (2) we shouldn't have proved it already in this course) and explain the proof to us. Your solution needs to include these three parts:

(a) Your source and some background for the application.

(b) Relate your proof to the theorem definition from lecture - what are the pigeons and what are the pigeonholes?

(c) What does the theorem tell us about them?

7.1. Some suggestions

We hope you'll think of something on your own! If you can't, here are some ideas that you can use:

- Husky Den Food Court options
- UW Course Conflicts (e.g., based on times of day; you can't do number of courses you take per quarter since we did that one in class).
- Olympic Sports (e.g., finish times of races)

7.2. Rubric information

These 'real world' problems are a bit different from other problems! Because of that we're showing you what our rubric is going to be for this first one so you can see our expectations.

(a) There should be enough information here that we can check that you have a correct application in the next part. This might be link(s) to website(s) where you got the information, or it might be you telling us how you found something (e.g., "I went to the foodcourt and counted options"). If it's something we might not understand (e.g., a hobby you think the grader might not share), explain enough for us to know what you're talking about.

(b) It needs to be clear what set your pigeons are and what your pigeonholes are.

(c) We need to understand the conclusion you're making, and it needs to be an accurate conclusion from part (b).

7.3. Sample Solution - Full credit

(a) Source: https://www.athletic.net/TrackAndField/meet/494374/results/m/1/1mile

Background: Nike Cross Nationals (NXN) is the defacto championship for high school running. There were 57 participants in the mile race this year. The range of the times (only counting whole seconds) was 27 seconds, with the champion running a 4:02 and the last-place finisher clocking in at 4:28.

(b) Let the pigeons be runners. Let the pigeonholes be mile times.

(c) Since there are 57 runners, $n = 57$. Since there were 27 different finishing times, $k = 27$. We observe that $\lceil \frac{57}{27} \rceil = 3$. Therefore, the pigeonhole principle allows us to conclude that at least one finishing time was attained by at least three people. This is confirmed by the results. In fact, 5 people finished with a 4:13. That must have been quite a fun race to watch.

8. Feedback [1 point]

Answer these questions on the separate Gradescope box for this question.

Please keep track of how much time you spend on this homework and answer the following questions. This can help us calibrate future assignments and future iterations of the course, and can help you identify which areas are most challenging for you.

- How many hours did you spend working on this assignment (excluding any extra credit questions, if applicable)? Report your estimate to the nearest hour.
- Which problem did you spend the most time on?
- Do you have any thoughts about specifically "Real-World: Is this a pigeon?" that you would like to share with us?
- Any other feedback for us?