CSE 312
Foundations of Computing II

Lecture 4: Intro to discrete probability

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Probability

- We want to model uncertainty.
  - i.e., outcome not determined a-priori
  - E.g. throwing dice, flipping a coin...
  - We want to numerically measure likelihood of outcomes = probability.
  - We want to make complex statements about these likelihoods.

- We will not argue why a certain physical process realizes the probabilistic model we study
  - Why is the outcome of the coin flip really “random”?

- First part of class: “Discrete” probability theory
  - Experiment with finite / discrete set of outcomes.
  - Will explore countably infinite and continuous outcomes later
Agenda

- Events
- Probability
- Equally Likely Outcomes
- Probability Axioms and Beyond Equally Likely Outcomes
- More Examples
Definition. A sample space \( \Omega \) is the set of all possible outcomes of an experiment.

Examples:

- Single coin flip: \( \Omega = \{H, T\} \)
- Two coin flips: \( \Omega = \{HH, HT, TH, TT\} \)
- Roll of a die: \( \Omega = \{1, 2, 3, 4, 5, 6\} \)
Events

**Definition.** An event $E \subseteq \Omega$ is a subset of possible outcomes.

**Examples:**
- Getting at least one head in two coin flips: $E = \{HH, HT, TH\}$
- Rolling an even number on a die: $E = \{2, 4, 6\}$
Events

**Definition.** An event \( E \subseteq \Omega \) is a subset of possible outcomes.

Examples:
- Getting at least one head in two coin flips: \( E = \{HH, HT, TH\} \)
- Rolling an even number on a die: \( E = \{2, 4, 6\} \)

**Definition.** Events \( E \) and \( F \) are **mutually exclusive** if \( E \cap F = \emptyset \) (i.e., can’t happen at same time)

Examples:
- For dice rolls: If \( E = \{2, 4, 6\} \) and \( F = \{1, 5\} \), then \( E \cap F = \emptyset \)
Suppose I roll two 4-sided dice. Let $D_1$ be the value of the blue die and $D_2$ be the value of the red die. To the right is the sample space (possible outcomes).

What outcomes match these events?

A. $D_1 = 1$

B. $D_1 + D_2 = 6$

C. $D_1 = 2 \times D_2$
Example: 4-sided Dice

Suppose I roll two 4-sided dice. Let $D_1$ be the value of the blue die and $D_2$ be the value of the red die. To the right is the sample space (possible outcomes).

What outcomes match these events?

A. $D_1 = 1$
   
   $A = \{(1,1), (1,2), (1,3), (1,4)\}$

B. $D_1 + D_2 = 6$
   
   $B = \{(2,4), (3,3), (4,2)\}$

C. $D_1 = 2 \times D_2$
   
   $C = \{(2,1), (4,2)\}$
Example: 4-sided Dice, Mutual Exclusivity

Are $A$ and $B$ mutually exclusive?
How about $B$ and $C$?

A. $D_1 = 1$

B. $D_1 + D_2 = 6$

C. $D_1 = 2 \times D_2$
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Idea: Probability

A **probability** is a number (between 0 and 1) describing how likely a particular outcome will happen.

Will define a function

\[ \mathbb{P}: \Omega \to [0, 1] \]

that maps outcomes \( \omega \in \Omega \) to probabilities.

– Also use notation: \( \mathbb{P}(\omega) = P(\omega) = \text{Pr}(\omega) \)
Example – Coin Tossing

Imagine we toss one coin – outcome can be heads or tails.

\[ \Omega = \{H, T\} \]

\[ \mathbb{P}\text{? } \text{Depends! What do we want to model?!} \]

**Fair coin toss**

\[ \mathbb{P}(H) = \mathbb{P}(T) = \frac{1}{2} = 0.5 \]
Example – Coin Tossing

Imagine we toss one coin – outcome can be heads or tails.

$$\Omega = \{H, T\}$$

$$\mathbb{P}?$$ Depends! What do we want to model?!

Bent coin toss (e.g., biased or unfair coin)

$$\mathbb{P}(H) = 0.45, \quad \mathbb{P}(T) = 0.55$$
Definition. A (discrete) probability space is a pair \((\Omega, \mathbb{P})\) where:

- \(\Omega\) is a set called the sample space.
- \(\mathbb{P}\) is the probability measure, a function \(\mathbb{P}: \Omega \rightarrow [0,1]\) such that:
  - \(\mathbb{P}(\omega) \geq 0\) for all \(\omega \in \Omega\)
  - \(\sum_{\omega \in \Omega} \mathbb{P}(\omega) = 1\)
**Definition.** A (discrete) **probability space** is a pair \((\Omega, \mathbb{P})\) where:

- \(\Omega\) is a set called the **sample space**.
- \(\mathbb{P}\) is the **probability measure**, a function \(\mathbb{P}: \Omega \to [0,1]\) such that:
  - \(\mathbb{P}(\omega) \geq 0\) for all \(\omega \in \Omega\)
  - \(\sum_{\omega \in \Omega} \mathbb{P}(\omega) = 1\)

Some outcome must show up

The likelihood (or probability) of each outcome is non-negative.

Either finite or infinite countable (e.g., integers)

Set of possible **elementary outcomes**

Specify Likelihood (or probability) of each elementary outcome
Uniform Probability Space

**Definition.** A **uniform probability space** is a pair $(\Omega, \mathbb{P})$ such that

$$\mathbb{P}(\omega) = \frac{1}{|\Omega|}$$

for all $\omega \in \Omega$.

Examples:

- Fair coin $P(\omega) = \frac{1}{2}$
- Fair 6-sided die $P(\omega) = \frac{1}{6}$

$$\sum_{\omega \in \Omega} P(\omega) = 1$$
Events

**Definition.** An event in a probability space \((\Omega, \mathbb{P})\) is a subset \(\mathcal{A} \subseteq \Omega\). Its probability is

\[
\mathbb{P}(\mathcal{A}) = \sum_{\omega \in \mathcal{A}} \mathbb{P}(\omega)
\]

Convenient abuse of notation: \(\mathbb{P}\) is extended to be defined over sets. \(\mathbb{P}(\omega) = \mathbb{P}(\{\omega\})\)
Agenda

• Events
• Probability
• **Equally Likely Outcomes**
• Probability Axioms and Beyond Equally Likely Outcomes
• More Examples
Think back to 4-sided die. Suppose each outcome is equally likely. What is the probability of event $B$? $\Pr(B) = \frac{3}{16}$

**Example: 4-sided Dice, Event Probability**

B. $D_1 + D_2 = 6$

$B = \{(2,4), (3,3)(4,2)\}$

\[
\Pr(B) = \sum_{\omega \in B} \Pr(\omega) = P(2,4) + P(3,3) + P(4,2) = \frac{3}{16}
\]
Equally Likely Outcomes

If \((\Omega, P)\) is a \textbf{uniform} probability space, then for any event \(E \subseteq \Omega\), then

\[
P(E) = \frac{|E|}{|\Omega|}
\]

This follows from the definitions of the prob. of an event and uniform probability spaces.
Example – Coin Tossing

Toss a coin 100 times. Each outcome is equally likely. What is the probability of seeing 50 heads?

\[ \mathcal{L} = \{ \text{sequences of 100 coin flips} \} \]
\[ \mathcal{L} = \{ H, T \}^{100} \]
\[ |\mathcal{L}| = 2^{100} \]
\[ \frac{|\mathcal{L}|}{2^{100}} = \frac{1}{2^{100}} \]

\[ \mathbb{P}(E) = \frac{|E|}{|\mathcal{L}|} \]

(A) $\frac{1}{2}$
(B) $\frac{1}{250}$
(C) $\frac{\binom{100}{50}}{2^{100}}$
(D) Not sure
Brain Break
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Axioms of Probability

Let $\Omega$ denote the sample space and $E, F \subseteq \Omega$ be events. Note this is applies to any probability space (not just uniform)

Axiom 1 (Non-negativity): $P(E) \geq 0$.

Axiom 2 (Normalization): $P(\Omega) = 1$.

Axiom 3 (Countable Additivity): If $E$ and $F$ are mutually exclusive, then $P(E \cup F) = P(E) + P(F)$.

Corollary 1 (Complementation): $P(E^c) = 1 - P(E)$.

Corollary 2 (Monotonicity): If $E \subseteq F$, $P(E) \leq P(F)$.

Corollary 3 (Inclusion-Exclusion): $P(E \cup F) = P(E) + P(F) - P(E \cap F)$. 

$\Omega = E \cup E^c$

$1 = P(\Omega) = P(E \cup E^c) = P(E) + P(E^c)$.
**Review Probability space**

**Definition.** A (discrete) **probability space** is a pair $(\Omega, \mathbb{P})$ where:

- $\Omega$ is a set called the **sample space**.
- $\mathbb{P}$ is the **probability measure**, a function $\mathbb{P}: \Omega \to [0,1]$ such that:
  - $\mathbb{P}(\omega) \geq 0$ for all $\omega \in \Omega$
  - $\sum_{\omega \in \Omega} \mathbb{P}(\omega) = 1$

Some outcome must show up

The likelihood (or probability) of each outcome is non-negative.

Either finite or infinite countable (e.g., integers)

Set of possible elementary outcomes

Specify Likelihood (or probability) of each elementary outcome

Ω
Non-equally Likely Outcomes

Probability spaces can have non-equally likely outcomes.

$\mathcal{N} = \{HH, HT, TH, TT\}$
More Examples of Non-equally Likely Outcomes
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• Events
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Example: Dice Rolls

Suppose I had a two, fair, 6-sided dice that we roll, one green, one red. What is the probability that we see at least one 3 in the two rolls.

\[ |\Sigma| = 36 \quad E = \{ \text{at least one 3} \} \]

\[ \Pr(E) = 1 - \Pr(E^c) \]

\[ \Pr(\text{see no 3's}) = \frac{25}{36} \]

\[ = \frac{11}{36} \]
Example: Birthday “Paradox”

Suppose we have a collection of \( n \) people in a room. What is the probability that at least 2 people share a birthday? Assume that there are exactly 365 possible birthdays, and each possibility is equally likely.

\[ S = \{ \text{Jan, Jan, Jan, Jan, Jan, Feb, Feb, Feb, Feb ...} \} \]

\[ |S| = 365^n \]

\[ \Pr(w) = \frac{1}{365^n} \]

\[ w = \{ \text{Jan-3, Mar 10, Feb 12, Apr 3 ...} \} \]

\[ E = \{ \text{at least 2 people share a bday} \} \]

\[ E^c = \{ \text{no two people share a bday} \} \]

\[ \Pr(E) = 1 - \Pr(E^c) \]
Example: Birthday “Paradox” cont.

\[ \Pr(E^c) = \frac{1-E^n}{1-E} = \frac{365 \cdot 364 \cdot 363 \cdots \cdot 365-(n-1)}{365^n} \]

\[ = \frac{365!}{(365-n)!} \cdot \frac{1}{365^n} \]

For \( n = 23 \), \( \Pr(E) > 0.5 \)

For \( n = 60 \), \( \Pr(E) > 0.99 \)

\[ 1 - \Pr(\text{none have Jan 1}) \]

\[ \frac{364^n}{365^n} \]
Example: Returning Homeworks

- Class with n students, randomly hand back homeworks. All permutations equally likely.

\[ \Omega = \text{perms of } n \text{ els} \]
\[ |\Omega| = n! \]
\[ P(\omega) = \frac{1}{n!} \]

- \( P(\text{person 17 gets their own hwback}) \)
  \[ = \frac{1}{n!} \]
\[
\frac{n!}{(n-1)!} = \frac{n}{n-1}\]

\[
\frac{\frac{1}{n^2}}{\frac{1}{n^3}} = \frac{n^3}{n^2} = n
\]