Midterm: Monday Feb 12, 1:30pm
Read my edstem post
Review Friday?

Locators:
ECE 125
EXED 110
GUG 220

Last Name
A-F
G-K
L-Z
Often we want to model experiments where the outcome is not discrete.
Example – Lightning Strike

Lightning strikes a pole within a one-minute time frame

• $T$ = time of lightning strike

• Every time within $[0,1]$ is equally likely
  – Time measured with infinitesimal precision.

The outcome space is not discrete
Lightning strikes a pole within a one-minute time frame

- $T$ = time of lightning strike
- Every point in time within $[0,1]$ is equally likely

$$P(T \geq 0.5) =$$
Lightning strikes a pole within a one-minute time frame

- $T$ = time of lightning strike
- Every point in time within $[0, 1]$ is equally likely

\[ P(0.2 \leq T \leq 0.5) = \]
Lightning strikes a pole within a one-minute time frame

- $T =$ time of lightning strike
- Every point in time within $[0,1]$ is equally likely

$P(T = 0.5) =$
Bottom line

- This gives rise to a different type of random variable
- \( P(T = x) = 0 \) for all \( x \in [0,1] \)
- Yet, somehow we want
  - \( P(T \in [0,1]) = 1 \)
  - \( P(T \in [a, b]) = b - a \)
  - ...
- How do we model the behavior of \( T \)?

First try: A discrete approximation
Example – Lightning Strike

Lightning strikes a pole within a one-minute time frame

- $X =$ time of lightning strike
- Every time within $[0,1]$ is equally likely
  - Time measured with infinitesimal precision.

Discrete approximation?
A Discrete Approximation

Probability Mass Function
PMF

\[ p_X \]

\[ \lim_{n \to \infty} \frac{i}{n} = \frac{1}{n} \quad i = 1, \ldots, n \]

\[ \sum_{n=1}^{\infty} \]
A Discrete Approximation

Probability Mass Function (PMF)

Cumulative Distribution Function (CDF)

\[ F_X(x) = P(X \leq x) \]

\[
F_X(x) = \begin{cases} 
0 & x < 0 \\
\sum_{i=1}^{n} \frac{p_i}{n} & 0 \leq x \leq 1 \\
1 & x > 1 
\end{cases}
\]
Recall: Cumulative Distribution Function (CDF)

\[ p_X(x) = P(X = x) \]

\[ \sum_{x \in \mathbb{X}} p_X(x) = 1 \quad p_X(x) \geq 0 \]

\[ F_X(x) = P(X \leq x) \]

\[ F_X \xrightarrow{\text{monotonic}} 0 \rightarrow 1 \]

\[ p_X(k) = \frac{F_X(k) - F_X(k-1)}{k - (k-1)} \]

\[ F_X(x) = \sum_{z \leq x} p_X(z) \]
\[ K = (1-c-1) \]

Let distance between values in \( S \rightarrow 0 \)

\[ f(x) = \frac{d}{dx} F_X(x) \]

Prob density for pdf

\[ F_X(x) = \int_{-\infty}^{x} f_X(x) \, dx \]

\[ P(X \leq x) \]

Capturing rate at which prob is accumulating at point \( x \)
Definition. A continuous random variable $X$ is defined by a probability density function (PDF) $f_X : \mathbb{R} \rightarrow \mathbb{R}$, such that

Non-negativity: $f_X(x) \geq 0$ for all $x \in \mathbb{R}$
Probability Density Function - Intuition

Non-negativity: \( f_X(x) \geq 0 \) for all \( x \in \mathbb{R} \)

Normalization: \( \int_{-\infty}^{+\infty} f_X(x) \, dx = 1 \)
Probability Density Function - Intuition

Non-negativity: \( f_X(x) \geq 0 \) for all \( x \in \mathbb{R} \)

Normalization: \( \int_{-\infty}^{+\infty} f_X(x) \, dx = 1 \)

\[
F(b) - F(a) = P(a \leq X \leq b) = \int_{a}^{b} f_X(x) \, dx
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Probability Density Function - Intuition

- **Non-negativity:** \( f_X(x) \geq 0 \) for all \( x \in \mathbb{R} \)

- **Normalization:** \( \int_{-\infty}^{+\infty} f_X(x) \, dx = 1 \)

\[
F(b) - F(a) = P(a \leq X \leq b) = \int_a^b f_X(x) \, dx
\]

\[
P(X = y) = P(y \leq X \leq y) = \int_y^y f_X(x) \, dx = 0
\]

- **Density ≠ Probability**

\[
f_X(y) \neq 0 \quad P(X = y) = 0
\]
Probability Density Function - Intuition

Non-negativity: $f_X(x) \geq 0$ for all $x \in \mathbb{R}$

Normalization: $\int_{-\infty}^{+\infty} f_X(x) \, dx = 1$

$F(b) - F(a) = P(a \leq X \leq b) = \int_{a}^{b} f_X(x) \, dx$

$P(X = y) = P(y \leq X \leq y) = \int_{y}^{y} f_X(x) \, dx = 0$

$P(X \approx y) \approx P\left(y - \frac{\epsilon}{2} \leq X \leq y + \frac{\epsilon}{2}\right) = \int_{y-\frac{\epsilon}{2}}^{y+\frac{\epsilon}{2}} f_X(x) \, dx \approx \epsilon f_X(y)$

What $f_X(x)$ measures: The local rate at which probability accumulates
Probability Density Function - Intuition

Non-negativity: \( f_X(x) \geq 0 \) for all \( x \in \mathbb{R} \)

Normalization: \( \int_{-\infty}^{+\infty} f_X(x) \, dx = 1 \)

\[
F(b) - F(a) = P(a \leq X \leq b) = \int_{a}^{b} f_X(x) \, dx
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P(X = y) = P(y \leq X \leq y) = \int_{y}^{y} f_X(x) \, dx = 0
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P(X \approx y) \approx P\left(y - \frac{\epsilon}{2} \leq X \leq y + \frac{\epsilon}{2}\right) = \int_{y-\frac{\epsilon}{2}}^{y+\frac{\epsilon}{2}} f_X(x) \, dx \approx \epsilon f_X(y)
\]

\[
\frac{P(X \approx y)}{P(X \approx z)} \approx \frac{\epsilon f_X(y)}{\epsilon f_X(z)} = \frac{f_X(y)}{f_X(z)}
\]
Definition. A **continuous random variable** \( X \) is defined by a probability density function (PDF) \( f_X: \mathbb{R} \rightarrow \mathbb{R} \), such that

**Non-negativity:** \( f_X(x) \geq 0 \) for all \( x \in \mathbb{R} \)

**Normalization:** \( \int_{-\infty}^{+\infty} f_X(x) \, dx = 1 \)

\[
F(b) - F(a) = P(a \leq X \leq b) = \int_{a}^{b} f_X(x) \, dx
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P(X = y) = P(y \leq X \leq y) = \int_{y}^{y} f_X(x) \, dx = 0
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\frac{P(X \approx y)}{P(X \approx z)} \approx \frac{\varepsilon f_X(y)}{\varepsilon f_X(z)} = \frac{f_X(y)}{f_X(z)}
\]
Cumulative Distribution Function

**Definition.** The cumulative distribution function (cdf) of $X$ is

$$F_X(a) = P(X \leq a) = \int_{-\infty}^{a} f_X(x) \, dx$$

By the fundamental theorem of Calculus

$$f_X(x) = \frac{d}{dx} F_X(x)$$
## From Discrete to Continuous

<table>
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<tr>
<th>Discrete</th>
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\[
 f_X(x) = \frac{d}{dx} F_X(x) 
\]
A Discrete Approximation

Probability Mass Function
PMF

Cumulative Distribution Function
CDF

$p_X$

$F_X$
\[
F_X(x) = \begin{cases} 
0 & x < 0 \\
0 & 0 \leq x \leq 1 \\
1 & x > 1 
\end{cases}
\]

\[
f_X(x) = \begin{cases} 
0 & x < 0 \\
1 & 0 \leq x \leq 1 \\
0 & x > 1 
\end{cases}
\]
PDF of Uniform RV

\( X \sim \text{Unif}(0,1) \)

\[
F_X(x) = P(X \leq x) = \begin{cases} 
0 & x \leq 0 \\
 x & 0 \leq x \leq 1 \\
1 & 1 \leq x 
\end{cases}
\]

\[
f_X(x) = \begin{cases} 
1, & x \in [0,1] \\
0, & x \notin [0,1] 
\end{cases}
\]
$X \sim \text{Unif}(0,1)$

Non-negativity: $f_X(x) \geq 0$ for all $x \in \mathbb{R}$

Normalization: $\int_{-\infty}^{+\infty} f_X(x) \, dx = 1$

$F(b) - F(a) = P(a \leq X \leq b) = \int_{a}^{b} f_X(x) \, dx$

$P(X = y) = P(y \leq X \leq y) = \int_{y}^{y} f_X(x) \, dx = 0$

$P(X \approx y) \approx \epsilon f_X(y)$

$\frac{P(X \approx y)}{P(X \approx z)} \approx \frac{\epsilon f_X(y)}{\epsilon f_X(z)} = \frac{f_X(y)}{f_X(z)}$

$f_X(x) = \begin{cases} 
1, & x \in [0,1] \\
0, & x \notin [0,1] 
\end{cases}$

$F_X(x) = P(X \leq x) = \begin{cases} 
0, & x \leq 0 \\
x, & 0 \leq x \leq 1 \\
1, & 1 \leq x
\end{cases}$
PDF of Uniform RV

\( X \sim \text{Unif}(0,1) \)

\[ f_X(x) = \begin{cases} 
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0, & x \notin [0,1] 
\end{cases} \]

Non-negativity: \( f_X(x) \geq 0 \) for all \( x \in \mathbb{R} \)

Normalization: \( \int_{-\infty}^{+\infty} f_X(x) \, dx = 1 \)

\[ \int_{-\infty}^{+\infty} f_X(x) \, dx = \int_{0}^{1} f_X(x) \, dx = 1 \cdot 1 = 1 \]
Probability of Event

$X \sim \text{Unif}(0,1)$

$f_X(x) = \begin{cases} 
1, & x \in [0,1] \\
0, & x \notin [0,1]
\end{cases}$

Non-negativity: $f_X(x) \geq 0$ for all $x \in \mathbb{R}$

Normalization: $\int_{-\infty}^{+\infty} f_X(x) \, dx = 1$

$P(a \leq X \leq b) = \int_{a}^{b} f_X(x) \, dx$

$a = 0.2$

$b = 0.5$

$\int 1 \, dx = x \bigg|_{0.2}^{0.5} = 0.3$
Probability of Event

\( X \sim \text{Unif}(0,1) \)

\[
f_X(x) = \begin{cases} 
1, & x \in [0,1] \\
0, & x \notin [0,1] 
\end{cases}
\]

Non-negativity: \( f_X(x) \geq 0 \) for all \( x \in \mathbb{R} \)

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P(X \approx y) \approx \epsilon f_X(y) = \epsilon
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\frac{P(X \approx y)}{P(X \approx z)} \approx \frac{\epsilon f_X(y)}{\epsilon f_X(z)} = \frac{f_X(y)}{f_X(z)}
\]
PDF of Uniform RV

\( X \sim \text{Unif}(0,0.5) \)

\[
f_X(x) = \begin{cases} 
0 & 0 \leq x \leq 0.5 \\
0 & \text{otherwise}
\end{cases}
\]

\[
\int C \, dx = 1
\]

\[
\begin{align*}
0.5 & = C x \bigg|_0^{0.5} \\
& = 0.5 C
\end{align*}
\]

\[
\Rightarrow C = 2
\]
PDF of Uniform RV

\( X \sim \text{Unif}(0, 0.5) \)

\[ f_X(x) = \begin{cases} 
2, & x \in [0, 0.5] \\
0, & x \notin [0, 0.5] 
\end{cases} \]

\[
\int_{-\infty}^{+\infty} f_X(x) \, dx = \int_0^1 f_X(x) \, dx = 2 \cdot 0.5 = 1
\]

Density ≠ Probability

\( f_X(x) \gg 1 \) is possible!

Probability on \([0, 0.5]\) accumulates at twice the rate compared to \(\text{Unif}(0,1)\).
PDF of Uniform RV

\[ X \sim \text{Unif}(0,0.5) \]

\[
f_X(x) = \begin{cases} 
2, & x \in [0,0.5] \\
0, & x \notin [0,0.5] 
\end{cases}
\]
Uniform Distribution

\[ X \sim \text{Unif}(a, b) \]

\[ f_X(x) = \begin{cases} 
\frac{1}{b - a} & x \in [a, b] \\
0 & \text{else}
\end{cases} \]

\[ \int_{-\infty}^{+\infty} f_X(x) \, dx = (b - a) \frac{1}{b - a} = 1 \]
Cumulative Distribution Function

Definition. The cumulative distribution function (cdf) of $X$ is

$$F_X(a) = P(X \leq a) = \int_{-\infty}^{a} f_X(x) \, dx$$

By the fundamental theorem of Calculus

$$f_X(x) = \frac{d}{dx} F_X(x)$$
## From Discrete to Continuous

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Cumulative Distribution Function

**Definition.** The cumulative distribution function (cdf) of $X$ is

$$F_X(a) = P(X \leq a) = \int_{-\infty}^{a} f_X(x) \, dx$$

By the fundamental theorem of Calculus $f_X(x) = \frac{d}{dx} F_X(x)$

Therefore: $P(X \in [a, b]) = F_X(b) - F_X(a)$

$F_X$ is monotone increasing, since $f_X(x) \geq 0$. That is $F_X(c) \leq F_X(d)$ for $c \leq d$

$$\lim_{a \to -\infty} F_X(a) = P(X \leq -\infty) = 0 \quad \lim_{a \to +\infty} F_X(a) = P(X \leq +\infty) = 1$$
Agenda

• Continuous Random Variables
• Probability Density Function
• Cumulative Distribution Function
• Expectation and Variance of continuous r.v.
• Introduction to continuous zoo
Definition. The expected value of a continuous RV $X$ is defined as

$$E[X] = \int_{-\infty}^{+\infty} f_X(x) \cdot x \, dx$$

Fact. $E[aX + bY + c] = aE[X] + bE[Y] + c$

Proof follows same ideas as discrete case
Expectation of a Continuous RV

**Definition.** The **expected value** of a continuous RV $X$ is defined as

$$
\mathbb{E}[X] = \int_{-\infty}^{+\infty} f_X(x) \cdot x \, dx
$$

**Fact.** $\mathbb{E}[aX + bY + c] = a\mathbb{E}[X] + b\mathbb{E}[Y] + c$

**Definition.** The **variance** of a continuous RV $X$ is defined as

$$
\text{Var}(X) = \int_{-\infty}^{+\infty} f_X(x) \cdot (x - \mathbb{E}[X])^2 \, dx = \mathbb{E}[X^2] - \mathbb{E}[X]^2
$$
Agenda

• Zoo of continuous random variables
  – Uniform Distribution
  – Exponential Distribution
  – Normal Distribution
Expectation of a Continuous RV

**Definition.**

\[ \mathbb{E}[X] = \int_{-\infty}^{+\infty} f_X(x) \cdot x \, dx \]

**Example.** \( T \sim \text{Unif}(0,1) \)

\[ f_T(x) = \begin{cases} 1, & x \in [0,1] \\ 0, & x \not\in [0,1] \end{cases} \]
Expectation of a Continuous RV

**Definition.**

\[ E[X] = \int_{-\infty}^{+\infty} f_X(x) \cdot x \, dx \]

**Example.** \( T \sim \text{Unif}(0,1) \)

\[ f_T(x) = \begin{cases} 1, & x \in [0,1] \\ 0, & x \notin [0,1] \end{cases} \]

\[ E[T] = \frac{1}{2} \cdot 1^2 = \frac{1}{2} \]

Area of triangle
Uniform Density – Expectation

\[ X \sim \text{Unif}(a, b) \]

\[
\mathbb{E}[X] = \int_{-\infty}^{+\infty} f_X(x) \cdot x \, dx
\]

\[
= \frac{1}{b-a} \int_{a}^{b} x \, dx = \frac{1}{b-a} \left( \frac{x^2}{2} \right) \bigg|_{a}^{b} = \frac{1}{b-a} \left( \frac{b^2 - a^2}{2} \right)
\]

\[
= \frac{(b-a)(a+b)}{2(b-a)} = \frac{a+b}{2}
\]

\[ f_X(x) = \begin{cases} 
\frac{1}{b-a} & x \in [a, b] \\
0 & \text{else}
\end{cases} \]
Uniform Density – Variance

\( X \sim \text{Unif}(a, b) \)

\[
\mathbb{E}[X^2] = \int_{-\infty}^{\infty} x^2 f_x(x) \, dx = \int_{a}^{b} x^2 \frac{1}{b-a} \, dx
\]

\[
f_x(x) = \begin{cases} 
\frac{1}{b-a} & x \in [a, b] \\
0 & \text{else}
\end{cases}
\]
Uniform Density – Variance

\( X \sim \text{Unif}(a, b) \)

\[
\mathbb{E}[X^2] = \int_{-\infty}^{+\infty} x^2 f_X(x) \, dx
\]

\[
= \frac{1}{b - a} \int_a^b x^2 \, dx = \frac{1}{b - a} \left( \frac{x^3}{3} \right) \bigg|_a^b = \frac{b^3 - a^3}{3(b - a)}
\]

\[
= \frac{(b - a)(b^2 + ab + a^2)}{3(b - a)} = \frac{b^2 + ab + a^2}{3}
\]

\[
f_X(x) = \begin{cases} 
1 & x \in [a, b] \\
\frac{1}{b - a} & \text{else}
\end{cases}
\]
Uniform Density – Variance

\[ X \sim \text{Unif}(a, b) \]

\[ \text{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 \]

\[ \mathbb{E}[X^2] = \frac{b^2 + ab + a^2}{3} \]

\[ \mathbb{E}[X] = \frac{a + b}{2} \]
Uniform Density – Variance

\[ X \sim \text{Unif}(a, b) \]

\[
\begin{align*}
\text{Var}(X) &= \mathbb{E}[X^2] - \mathbb{E}[X]^2 \\
&= \frac{b^2 + ab + a^2}{3} - \frac{a^2 + 2ab + b^2}{4} \\
&= \frac{4b^2 + 4ab + 4a^2}{12} - \frac{3a^2 + 6ab + 3b^2}{12} \\
&= \frac{b^2 - 2ab + a^2}{12} = \frac{(b - a)^2}{12}
\end{align*}
\]

\[
\begin{align*}
\mathbb{E}[X^2] &= \frac{b^2 + ab + a^2}{3} \\
\mathbb{E}[X] &= \frac{a + b}{2}
\end{align*}
\]
Uniform Distribution Summary

\( X \sim \text{Unif}(a, b) \)

\[
f_X(x) = \begin{cases} 
\frac{1}{b - a} & x \in [a, b] \\
0 & \text{else}
\end{cases}
\]

\[
F_X(y) = \begin{cases} 
0 & x < a \\
\frac{x - a}{b - a} & x \in [a, b] \\
1 & x > b
\end{cases}
\]

\[
\mathbb{E}[X] = \frac{a + b}{2}
\]

\[
\text{Var}(X) = \frac{(b - a)^2}{12}
\]
Agenda

• Zoo of continuous random variables
  – Uniform Distribution
  – Exponential Distribution
  – Normal Distribution
Exponential Density

Assume expected # of occurrences of an event per unit of time is $\lambda$ (independently)

- Cars going through intersection
- Number of lightning strikes
- Requests to web server
- Patients admitted to ER

Numbers of occurrences of event in one unit of time: Poisson distribution

$$P(W = i) = e^{-\lambda} \frac{\lambda^i}{i!}$$

(Discrete)

How long to wait until next event? Exponential density!

Let’s define it and then derive it!
Exponential Density - Warmup

Assume expected # of occurrences of an event per unit of time is $\lambda$ (independently)

What is $E[Z_t]$ where $Z_t = \#$ occurrences of event per $t$ units of time?

$E(Z_1) = \lambda$
$E(Z_3) = 3\lambda$
$E(Z_{0.2}) = 0.2\lambda$

$Z_t \sim \text{Poisson} (\lambda t)$
Exponential Density - Warmup

Assume expected number of occurrences of an event per unit of time is $\lambda$ (independently)

What is the distribution of $Z_t = \#$ occurrences of event per $t$ units of time?

\[ \mathbb{E}[Z_t] = t\lambda \]

$Z_t$ is independent over disjoint intervals

So $Z_t \sim Poi(t\lambda)$
The Exponential PDF/CDF

Assume expected # of occurrences of an event per unit of time is $\lambda$ (independently)

Numbers of occurrences of event: Poisson distribution

How long to wait until next event? Exponential density!

- Let $X$ be the time till the first event. We will compute $F_X(t)$ and $f_X(t)$
- We know $Z_t \sim Poi(t\lambda)$ is the # of events in the first $t$ units of time, for $t \geq 0$.

$$P(X > t) = P(Z_+ = 0) = e^{-\lambda t} \frac{(\lambda t)^0}{0!} = e^{-\lambda t}$$

$$F_X(t) = P(X \leq t) = 1 - e^{-\lambda t}$$

$$f_X(t) = \frac{d}{dt} F_X(t) = 2e^{-\lambda t} \quad t > 0$$
The Exponential PDF/CDF

Assume expected # of occurrences of an event per unit of time is $\lambda$ (independently)

Numbers of occurrences of event: Poisson distribution

How long to wait until next event? Exponential density!

- The exponential RV has range $[0, \infty)$, unlike Poisson with range $\{0,1,2, ... \}$
- Let $X \sim \text{Exp}(\lambda)$ be the time till the first event. We will compute $F_X(t)$ and $f_X(t)$
- We know $Z_t \sim \text{Poi}(t\lambda)$ be the # of events in the first $t$ units of time, for $t \geq 0$.
- $P(X > t) = P(\text{no event in the first } t \text{ units}) = P(Z_t = 0) = e^{-t\lambda} \frac{(t\lambda)^0}{0!} = e^{-t\lambda}$
- $F_X(t) = P(X \leq t) = 1 - P(Y > t) = 1 - e^{-t\lambda}$
- $f_X(t) = \frac{d}{dt} F_X(t) = \lambda e^{-t\lambda}$
**Exponential Distribution**

**Definition.** An exponential random variable $X$ with parameter $\lambda \geq 0$ is follows the exponential density

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

We write $X \sim \text{Exp}(\lambda)$ and say $X$ that follows the exponential distribution.

**CDF:** For $y \geq 0$,

$$F_X(y) = 1 - e^{-\lambda y}$$

Graph showing the exponential density for different values of $\lambda$: $\lambda = 2$, $\lambda = 1.5$, $\lambda = 1$, $\lambda = 0.5$. The CDF curves decrease as $\lambda$ decreases, indicating a faster decay for smaller values of $\lambda$. The graph also shows how the area under the curve equals 1 for all $y \geq 0$. The cumulative distribution function (CDF) represents the probability that a random variable is less than or equal to a specific value. For exponential distributions, the CDF is given by $1 - e^{-\lambda y}$.
Expectation

\[
\mathbb{E}[X] = \int_{-\infty}^{+\infty} f_X(x) \cdot x \, dx
\]

\[f_X(x) = \begin{cases} 
\lambda e^{-\lambda x} & x \geq 0 \\
0 & x < 0 \end{cases}\]

\[P(X > t) = e^{-t\lambda}\]
Expectation

\[ \mathbb{E}[X] = \int_{-\infty}^{+\infty} f_X(x) \cdot x \, dx \]
\[ = \int_{0}^{+\infty} \lambda e^{-\lambda x} \cdot x \, dx \]
\[ = \left( -(x + \frac{1}{\lambda})e^{-\lambda x} \right)_0^\infty = \frac{1}{\lambda} \]

Somewhat complex calculation
use integral by parts

\[ f_X(x) = \begin{cases} 
\lambda e^{-\lambda x} & x \geq 0 \\
0 & x < 0 
\end{cases} \]

\[ P(X > t) = e^{-t\lambda} \]

\[ \mathbb{E}[X] = \frac{1}{\lambda} \]

\[ \text{Var}(X) = \frac{1}{\lambda^2} \]
Exponential Distribution

We write $X \sim \text{Exp}(\lambda)$ and say $X$ that follows the exponential distribution.

**Definition.** An exponential random variable $X$ with parameter $\lambda \geq 0$ is follows the exponential density

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

**CDF:** For $y \geq 0$,

$$F_X(y) = 1 - e^{-\lambda y}$$

**$E[X]$**

$$E[X] = \frac{1}{\lambda}$$

**Var($X$)**

$$\text{Var}(X) = \frac{1}{\lambda^2}$$

$$P(X > t) = e^{-t\lambda}$$
Memorylessness

**Definition.** A random variable is **memoryless** if for all $s, t > 0$,

$$P(X > s + t \mid X > s) = P(X > t).$$

**Fact.** $X \sim \text{Exp}(\lambda)$ is memoryless.

Assuming an exponential distribution, if you’ve waited $s$ minutes, the probability of waiting $t$ more is exactly the same as when $s = 0$. 
Memorylessness of Exponential

Fact. \( X \sim \text{Exp}(\lambda) \) is memoryless.

Proof.

\[
P(X > s + t \mid X > s) =
\]

Proof that assuming exp distr, if you’ve waited \( s \) minutes, prob of waiting \( t \) more is exactly same as when \( s = 0 \)

\[
P(X > t) = e^{-\lambda t}
\]
**Memorylessness of Exponential**

**Fact.** $X \sim \text{Exp}(\lambda)$ is memoryless.

**Proof.**

\[
P(X > s + t \mid X > s) = \frac{P(\{X > s + t\} \cap \{X > s\})}{P(X > s)}
\]

\[
= \frac{P(X > s + t)}{P(X > s)}
\]

\[
= e^{-\lambda(s+t)} = e^{-\lambda t} = P(X > t)
\]

The only memoryless RVs are the geometric RV (discrete) and Exp RV (continuous).
Example

- Time it takes to check someone out at a grocery store is exponential with an expected value of 10 mins.
- Independent for different customers
- If you are the second person in line, what is the probability that you will have to wait between 10 and 20 mins?
Example

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- Independent for different customers
- If you are the second person in line, what is the probability that you will have to wait between 10 and 20 mins?

\[
T \sim \text{Exp}\left(\frac{1}{10}\right)
\]

\[
P(10 \leq T \leq 20) = \int_{10}^{20} \frac{1}{10} e^{-\frac{x}{10}} \, dx
\]

\[
y = \frac{x}{10} \text{ so } dy = \frac{dx}{10}
\]

\[
P(10 \leq T \leq 20) = \int_{1}^{2} e^{-y} \, dy = -e^{-y} \Big|_{1}^{2} = e^{-1} - e^{-2}
\]
Example

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- Independent for different customers
- If you are the second person in line, what is the probability that you will have to wait between 10 and 20 mins?

\[ T \sim \text{Exp}\left(\frac{1}{10}\right) \]

so \( F_T(t) = 1 - e^{-\frac{t}{10}} \)

\[ P(10 \leq T \leq 20) = F_T(20) - F_T(10) = 1 - e^{-\frac{20}{10}} - \left(1 - e^{-\frac{10}{10}}\right) = e^{-1} - e^{-2} \]
Agenda

- Zoo
  - Uniform Distribution
  - Exponential Distribution
  - Normal Distribution
The Normal Distribution

**Definition.** A Gaussian (or normal) random variable with parameters $\mu \in \mathbb{R}$ and $\sigma \geq 0$ has density

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

We say that $X$ follows the Normal Distribution, and write $X \sim \mathcal{N}(\mu, \sigma^2)$. 

$\mathcal{N}(0, 1)$. 

Carl Friedrich Gauss
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Proof of expectation is easy because density curve is symmetric around $\mu$,

$$f_X(\mu - x) = f_X(\mu + x),$$

but proof for variance requires integration of $e^{-x^2/2}$

We will see next time why the normal distribution is (in some sense) the most important distribution.
The Normal Distribution

Aka a “Bell Curve” (imprecise name)