CSE 312
Section 4
Random Variables
PSet 2 grades released
(regrade requests open ~24 hours after grades are released and close after a week)

PSet 3
(written & coding part due yesterday)

PSet 4 is released! Start early :D
Review
An outcome from a random experiment

Some number (the range of X is the set of possible values X can take on)

X
a random variable

Probability Mass Function (PMF)

\[ P(X=k) \]

probability that the random variable X will take on the value k

what is the probability of an outcome that will result in X being k

for discrete random variables (random variables with a finite, countably infinite range), this may sometimes be a piecewise function
Random variable
Captures a *quantitative property* (some numerical value that describes the outcome) of the outcome in a random experiment

e.g., sum of the dices on an random experiment where we roll 2 dice
Random Variables

An outcome from a random experiment

\( X \)

a random variable

Some number

(probability that the random variable \( X \) will take on the value \( k \))

probability that \( X \) will take on the value \( k \)

some number

(range or support) of \( X \)

(set of possible values \( X \) can take on)

Probability Mass Function (PMF)

\[ p_X(k) = P(X=k) \]

what is the probability of an outcome that will result in \( X \) being \( k \)

for discrete random variables (random variables with a finite, countably infinite range), this may sometimes be a piecewise function
Random Variables

An outcome from a random experiment

$X$ a random variable

Cumulative Distribution Function

$F_X(k) = P(X \leq k)$ — probability that the value $X$ takes on is less than or equal to $k$

Some number (the range (or support) of $X$ (sometimes denoted as $\Omega_X$) is the set of possible values $X$ can take on)

what is the probability of an outcome that will result in $X$ being $\leq k$

often can be derived from the PDF
Random Variables

An outcome from a random experiment

\(X\)

a random variable

Some number

(\text{the range (or support) of } X
\text{ (sometimes denoted as } \Omega_X\text{) is the set of possible values } X \text{ can take on})

Expectation

\[ E[X] = \sum (k \cdot P(X=k)) \]

sum of values in the range of \(X\), weighted by the probability

on average, what value can we “expect” \(X\) to take?

think about it like a weighted average of all the possible values \(X\) could be (weighted by the \(P(X=k)\))
Random Variables

An outcome from a random experiment

$X$ a random variable

Some number
(sometimes denoted as $\Omega_X$)

is
the set of possible values $X$ can take on)

Expectation

$$E[X] = \sum (k \cdot P(X=k))$$

sum of values in the range of $X$, weighted by the probability

on average, what value can we “expect” $X$ to take?

just averaging all the possible values of $X$ wouldn’t work since each outcome isn’t necessarily equally likely
Random Variables

An outcome from a random experiment

A \textit{random variable} \( X \)

Some number (the range (or support) of \( X \) (sometimes denoted as \( \Omega_X \)) is the set of possible values \( X \) can take on)

\textbf{Expectation of a function of } X \textbf{ (aka “Law of the Unconscious Statistician” (aka “LOTUS”))}

\[ E[f(X)] = \sum (f(k) \cdot P(X=k)) \]

\textit{On average, what value can we “expect” } f(X) \textit{ to take?}
Linearity of Expectation is a powerful property of random variables!
Random Variables
allow us to represent a quantitative property of a random experiment

**EXPECTATION** - weighted average of possible outcomes

you could use “brute force” and use the formula for expectation \( E[X] = \sum (x \cdot P(x)) \)

*sometimes, just applying the formula can be messy, so LoE comes in handy*

**LINEARITY OF EXPECTATION (LoE)** is one important property

\[
E(X+Y) = E(X) + E(Y)
\]

the expected value of the sum of 2 random variables is
the sum of their expected values
02 - Linearity of Expectation

| E(X+Y) | = | E(X) + E(Y) |

The expected value of the sum of 2 random variables is the sum of their expected values.

*This gives us a helpful tool to calculate expectations of complex RVs.*
This gives us a helpful tool to calculate expectations of complex RVs. The expected value of the sum of two random variables is the sum of their expected values:

\[ E(X+Y) = E(X) + E(Y) \]

This can be decomposed into a sum of random variables:

\[ X = X_1 + X_2 + \ldots + X_n \]
This gives us a helpful tool to calculate expectations of complex RVs. Decompose into a sum of random variables:

\[ X = X_1 + X_2 + \ldots + X_n \]

Apply linearity of expectation:

\[ E[X] = E[X_1] + E[X_2] + \ldots + E[X_n] \]
this gives us a helpful tool to calculate expectations of complex RVs

**DECOMPOSE** into a sum of random variables

\[ X = X_1 + X_2 + \ldots + X_n \]

**APPLY** linearity of expectation

\[ E[X] = E[X_1] + E[X_2] + \ldots + E[X_n] \]

**CONQUER** and calculate each value

\[ E[X_1] = \ldots, \ E[X_2] = \ldots, \ldots \]

\[ E(X+Y) = E(X) + E(Y) \]

the expected value of the sum of 2 random variables is

the sum of their expected values
**02 - Linearity of Expectation**

The expected value of the sum of two random variables, \( E(X+Y) = E(X) + E(Y) \), is the sum of their expected values.

Sometimes, these \( X_i \) variables we "decompose" \( X \) into are indicator random variables.

This gives us a helpful tool to calculate expectations of complex RVs:

- **DECOMPOSE** into a sum of random variables: \( X = X_1 + X_2 + \ldots + X_n \)
- **APPLY** linearity of expectation: \( E[X] = E[X_1] + E[X_2] + \ldots + E[X_n] \)
- **CONQUER** and calculate each value: \( E[X_1] = \ldots, E[X_2] = \ldots, \ldots \)
02 - Linearity of Expectation

**DECOMPOSE** into a sum of random variables

**APPLY** linearity of expectation

**CONQUER** and calculate each value

\[ X = X_1 + X_2 + \ldots + X_n \]

\[ E[X] = E[X_1] + E[X_2] + \ldots + E[X_n] \]

sometimes, these \( X_i \) variables we “decompose” \( X \) into are **indicator** random variables

**this gives us a helpful tool to calculate expectations of complex RVs**

***\( X \) and \( Y \) DON’T have to be independent!***

\[ E(X + Y) = E(X) + E(Y) \]

the expected value of the sum of 2 random variables is the sum of their expected values

\[ E[X] = \ldots, \ E[X_2] = \ldots, \ldots \]
Indicator Random Variables

We can define an \textit{indicator random variable} \( X \) for an event \( A \)

\[
X = \begin{cases} 
1 & \text{if event } A \text{ happens} \\
0 & \text{if event } A \text{ doesn’t happen}
\end{cases}
\]

\( X \) tells us whether event \( A \) will happen \( \rightarrow \) so, \( P(X = 1) = P(A) \)

\textit{Note that} \( E[X] = 1 \times P(X=1) + 0 \times P(X=0) = P(X=1) \)

This is why indicator RVs can be really useful when applying linearity of expectation!
Additional slides for content that will be covered later in the week!
Linearity of expectation is special!

\[ E[X+Y] = E[X] + E[Y] \quad \text{but} \quad E[X^2] \neq (E[X])^2 \]

Instead...

\[ E[g(X)] = \sum (g(x) \times P(X=x)) \]
Variance

Variance is another property of RVs (like expectation) that measures how much the values in the RV "vary"
Random Variables allow us to represent a quantitative property of a random experiment

**VARIANCE** - how “different” are values from the expectation “on average”

every random variable has some variance

\[
\text{Var}(X) = E[(X - E(X))^2] = \sum_x (P(X=x) \cdot (x - E(X))^2)
\]

- expected value of the squared distance between each RV outcome and the expected value of RV
- add up all the squared distances weighted by their probabilities

variance = (standard deviation)^2
Random Variables
allow us to represent a quantitative property of a random experiment

**VARIANCE** - how “different” are values from the expectation “on average”

every random variable has some variance

\[
\text{Var}(X) = \mathbb{E}[ (X - \mathbb{E}(X))^2 ] = \sum_x (\mathbb{P}(X=x) \times (x - \mathbb{E}(X))^2)
\]

**Properties**

\[
\begin{align*}
\text{Var}(a \cdot X + b) &= a^2 \cdot \text{Var}(X) \\
\text{Var}(X) &= \mathbb{E}[X^2] - (\mathbb{E}[X])^2
\end{align*}
\]
Random Variables allow us to represent a quantitative property of a random experiment.

**VARIANCE** - how “different” are values from the expectation “on average”

Every random variable has some variance.

\[
\text{Var}(X) = \mathbb{E} \left[ (X - \mathbb{E}(X))^2 \right] = \sum_x (P(X=x) \cdot (x - \mathbb{E}(X))^2)
\]

**Properties**

\[
\text{Var}(a \cdot X + b) = a^2 \cdot \text{Var}(X) \\
\text{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2
\]
What does independence mean for random variables?
Random variables $X$ and $Y$ are independent if, for all $x, y$ in the ranges of $X$ and $Y$:

$$P(X=x, \ Y=y) = P(X=x) \cdot P(Y=y)$$

Knowing the value of $X$ doesn’t help “guess” what $Y$ is.
Random variables $X$ and $Y$ are independent if

$$P(X=x, Y=y) = P(X=x) \cdot P(Y=y)$$

Knowing the value of $X$ doesn’t help “guess” what $Y$ is

It’s a useful property! If $X$ and $Y$ are independent random variables then

$$E(X \cdot Y) = E[X] \cdot E[Y]$$

$$Var(X + Y) = Var[X] + Var[Y] \quad \text{Linearity of variance holds}$$
Random variables $X$ and $Y$ are independent if –

$$P(X=x, Y=y) = P(X=x) \cdot P(Y=y)$$

Knowing the value of $X$ doesn’t help “guess” what $Y$ is.

Additionally, there’s independent and identically distributed (aka, “i.i.d.”) random variables.

In addition to independence, i.i.d. random variables also have the same pmf.

For example, rolling a die twice, where $X$ is the first roll number and $Y$ is the second roll number.
Problems!