

# CSE 312

## Section 4

Random Variables





## 01 - Reminders!



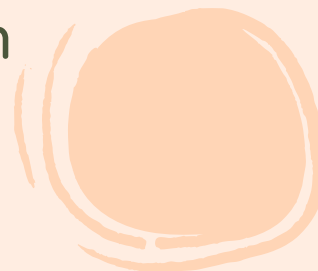
### HW 2 grades released

*(regrade requests open ~24 hours after grades are released and close after a week)*

### HW 3

*(written & coding part due yesterday)*

Midterm on July 15 @ 3:30 - 5:20 in  
BAG131





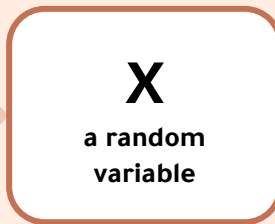
Review



## 02 - Random Variables



An outcome  
from a random  
experiment



Some number  
(the **range of X** is the set of  
possible values X can take on)



**Probability Mass Function (PMF)**

**$P(X=k)$**

probability that the random variable **X** will take on  
the value **k**

*what is the probability of an outcome that will result in X being k*

for discrete random variables (random variables with a finite, countably infinite range), this may sometimes be a piecewise function



# Random Variables



01

02

03

04



## Random variable

Captures a quantitative property  
(some numerical value that describes the  
outcome) of the outcome in a random  
experiment

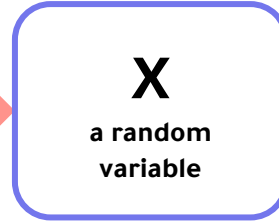
*e.g., sum of the dices on an random  
experiment where we roll 2 dice*



# Random Variables



An outcome  
from a random  
experiment



Some number  
(the **range (or support)** of **X**  
(sometimes denoted as  $\Omega_X$ ) is  
the set of possible values **X** can  
take on)

## Probability Mass Function (PMF)

$$p_X(k) = P(X=k)$$

probability that the random  
variable **X** will take on the value  
**k**

*what is the probability of an outcome  
that will result in **X** being **k***

for discrete random variables (random variables with a finite, countably infinite range), this may sometimes be a piecewise function



01

02

03

04

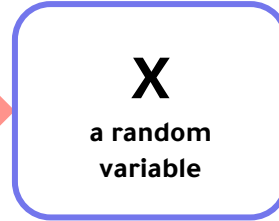




# Random Variables



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## Cumulative Distribution Function

$$F_X(k) = P(X \leq k) \rightarrow \text{probability that the value } X \text{ takes on is less than or equal to } k$$

*what is the probability of an outcome that will result in X being  $\leq k$*

often can be derived from the PDF



01

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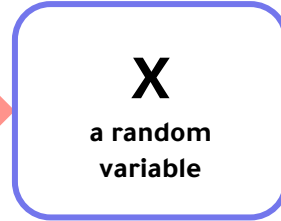




# Random Variables



An outcome  
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experiment



Some number  
(the **range (or support)** of **X**  
(sometimes denoted as  $\Omega_X$ ) is  
the set of possible values **X** can  
take on)

Expectation

$$E[X] = \sum(k \cdot P(X=k))$$

sum of values in the range of **X**,  
weighted by the probability  
*on average, what value can we "expect" **X** to take?*

think about it like a weighted average of all the possible values **X** could be (weighted by the  $P(X=k)$ )



01

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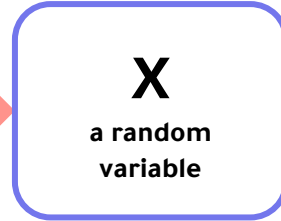




# Random Variables



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Expectation

$$E[X] = \sum(k \cdot P(X=k))$$

sum of values in the range of **X**,  
weighted by the probability  
*on average, what value can we "expect" **X** to take?*

just averaging all the possible values of **X** wouldn't work since each outcome isn't necessarily equally likely



01

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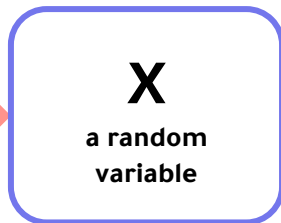




# Random Variables



An outcome  
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Some number  
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take on)

Expectation of a function of **X** (aka “Law of the Unconscious Statistician”  
(aka “LOTUS”))

$$E[f(X)] = \sum (f(k) \cdot P(X=k))$$

(note that the  
probabilities are  
still weighted  
using **X** (not **f(X)**)

*on average, what value can we “expect” **f(X)** to take?*



01

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03

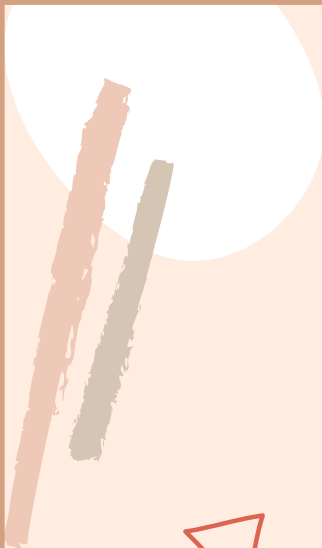
04





# LoE

Linearity of **E**xpectation is a powerful property of random variables!





## 02 - Linearity of Expectation



### Random Variables

*allow us to represent a quantitative property of a random experiment*

**EXPECTATION** - weighted average of possible outcomes

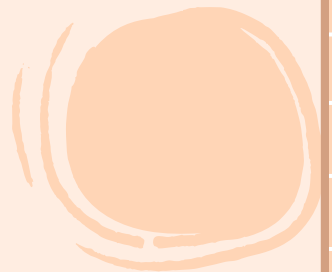
you could use “brute force” and use the formula for expectation ( $E[X] = \sum (x * P(x))$ )

*sometimes, just applying the formula can be messy, so LoE comes in handy*

**LINEARITY OF EXPECTATION (LoE)** *is one important property*

$$E(X+Y) = E(X) + E(Y)$$

the expected value of the sum of 2 random variables is  
the sum of their expected values





## 02 - Linearity of Expectation

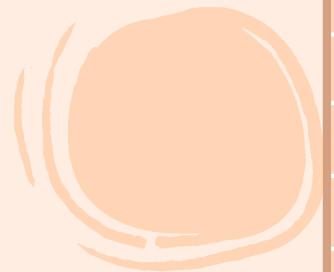


$$\mathbf{E(X+Y) = E(X) + E(Y)}$$

the expected value of the sum of 2 random variables is  
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---

*this gives us a helpful **tool to calculate expectations of complex RVs***





## 02 - Linearity of Expectation



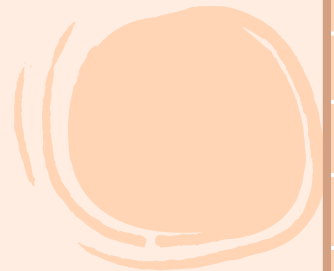
$$E(X+Y) = E(X) + E(Y)$$

the expected value of the sum of 2 random variables is  
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*this gives us a helpful **tool to calculate expectations of complex RVs***

**DECOMPOSE** into a sum of random variables

$$X = X_1 + X_2 + \dots + X_n$$





## 02 - Linearity of Expectation



$$E(X+Y) = E(X) + E(Y)$$

the expected value of the sum of 2 random variables is  
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*this gives us a helpful **tool to calculate expectations of complex RVs***

**DECOMPOSE** into a sum of random variables

$$X = X_1 + X_2 + \dots + X_n$$

**APPLY** linearity of expectation

$$E[X] = E[X_1] + E[X_2] + \dots + E[X_n]$$



## 02 - Linearity of Expectation



$$E(X+Y) = E(X) + E(Y)$$

the expected value of the sum of 2 random variables is the sum of their expected values

*this gives us a helpful **tool to calculate expectations of complex RVs***

**DECOMPOSE** into a sum of random variables  $X = X_1 + X_2 + \dots + X_n$

**APPLY** linearity of expectation  $E[X] = E[X_1] + E[X_2] + \dots + E[X_n]$

**CONQUER** and calculate each value  $E[X_1] = \dots, E[X_2] = \dots, \dots$



$$E(X+Y) = E(X) + E(Y)$$

the expected value of the sum of 2 random variables is the sum of their expected values

sometimes, these  $X_i$  variables we "decompose"  $X$  into are **indicator** random variables

*this gives us a helpful tool to calculate expectations of complex RVs*

DECOMPOSE into a sum of random variables

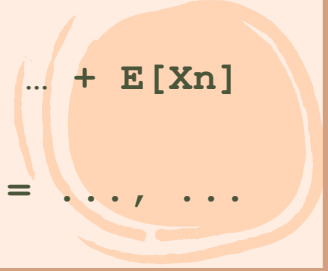
$$X = X_1 + X_2 + \dots + X_n$$

APPLY linearity of expectation

$$E[X] = E[X_1] + E[X_2] + \dots + E[X_n]$$

CONQUER and calculate each value

$$E[X_1] = \dots, E[X_2] = \dots, \dots$$





## 02 - Linearity of Expectation



\*\*\*X and Y DON'T have to be independent!

$$E(X+Y) = E(X) + E(Y)$$

the expected value of the sum of 2 random variables is the sum of their expected values

sometimes, these  $X_i$  variables we "decompose"  $X$  into are **indicator** random variables

this gives us a helpful **tool to calculate expectations of complex RVs**

DECOMPOSE into a sum of random variables

$$X = X_1 + X_2 + \dots + X_n$$

APPLY linearity of expectation

$$E[X] = E[X_1] + E[X_2] + \dots + E[X_n]$$

CONQUER and calculate each value

$$E[X_1] = \dots, E[X_2] = \dots, \dots$$



## 02 - Linearity of Expectation



### Indicator Random Variables

we can define a *indicator random variable*  $X$  for an event  $A$

$$X = \begin{cases} 1 & \text{if event } A \text{ happens} \\ 0 & \text{if event } A \text{ doesn't happen} \end{cases}$$

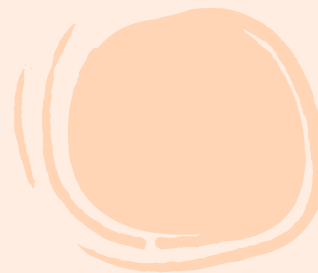
$X$  tells us whether event  $A$  will happen  $\rightarrow$  so,  $P(X = 1) = P(A)$

Note that  $E[X] = 1 * P(X=1) + 0 * P(X=0) = P(X=1)$

*this is why indicator RVs  
can be really useful when  
applying linearity of  
expectation!*



**Additional slides for content that  
will be covered later in the week!**





## 02 - Linearity of Expectation



**linearity of expectation is special!**

$$E[X+Y] = E[X] + E[Y] \text{ but } E[X^2] \neq (E[X])^2$$

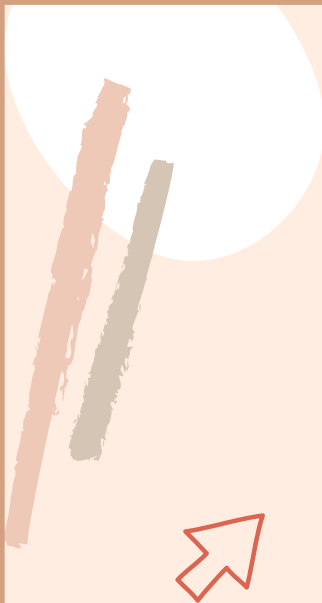
*instead...*

$$E[g(X)] = \sum (g(x) * P(X=x))$$



# Variance

Variance is a another property of RVs (like expectation) that measures how much the values in the RV “vary”





## 03 - Variance



### Random Variables

*allow us to represent a quantitative property of a random experiment*

**VARIANCE** - how “different” are values from the expectation “on average”

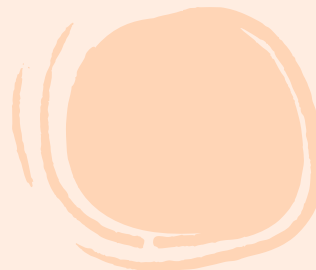
*every random variable has some variance*

$$\text{Var}(X) = E[(X - E(X))^2] = \sum_x (P(X=x) * (x - E(X))^2)$$

expected value of the squared distance between each RV outcome and the expected value of RV

add up all the squared distances weighted by their probabilities

$$\text{variance} = (\text{standard deviation})^2$$





## 03 - Variance



### Random Variables

*allow us to represent a quantitative property of a random experiment*

**VARIANCE** - how “different” are values from the expectation “on average”

*every random variable has some variance*

$$\text{Var}(X) = E[(X - E(X))^2] = \sum_x (P(X=x) * (x - E(X))^2)$$

#### *Properties*

$$\text{Var}(a \cdot X + b) = a^2 \cdot \text{Var}(X)$$

$$\text{Var}(X) = E[X^2] - (E[X])^2$$







## 03 - Variance



### Random Variables

*allow us to represent a quantitative property of a random experiment*

**VARIANCE** - how “different” are values from the expectation “on average”

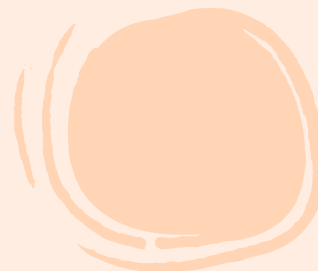
*every random variable has some variance*

$$\text{Var}(X) = E[(X - E(X))^2] = \sum_x (P(X=x) * (x - E(X))^2)$$

#### *Properties*

$$\text{Var}(a \cdot X + b) = a^2 \cdot \text{Var}(X)$$

$$\text{Var}(X) = E[X^2] - (E[X])^2$$





# INDEPENDENT RV

What does independence mean for  
random variables?





## Random Variable Independence



Random variables  $X$  and  $Y$  are **independent** if, for all  $x, y$  in the ranges of  $X$  and  $Y$ :

$$P(X=x, Y=y) = P(X=x) \cdot P(Y=y)$$

*Knowing the value of  $X$  doesn't help "guess" what  $Y$  is*





## Random Variable Independence



Random variables  $X$  and  $Y$  are **independent** if –

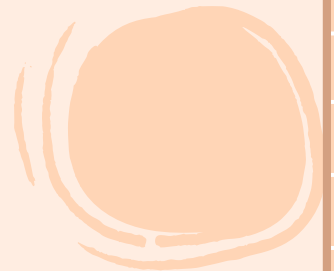
$$P(X=x, Y=y) = P(X=x) \cdot P(Y=y)$$

*Knowing the value of  $X$  doesn't help "guess" what  $Y$  is*

it's a useful property! if  $X$  and  $Y$  are independent random variables then –

$$E(X \cdot Y) = E[X] \cdot E[Y]$$

$$\text{Var}(X + Y) = \text{Var}[X] + \text{Var}[Y] \quad \textit{Linearity of variance holds}$$





## Random Variable Independence



Random variables  $X$  and  $Y$  are **independent** if –

$$P(X=x, Y=y) = P(X=x) \cdot P(Y=y)$$

*Knowing the value of  $X$  doesn't help "guess" what  $Y$  is*

Additionally, there's **independent and identically distributed** (aka, "i.i.d.") random variables

In addition to independence, i.i.d. random variables also **have the same pmf.**

For example, rolling a die twice, where  $X$  is the first roll number and  $Y$  is the second roll number





Problems!