CSE 312
Section 4

Random Variables

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01 - Reminders!

HW 2 grades released

(regrade requests open ~24 hours after grades are released and close after a week)

HW 3

(written & coding part due yesterday)

Midterm on July 15 @ 3:30 - 5:20 in BAG131

Review

02 - Random Variables

An outcome from a random experiment → **X** **a random variable** → *Some number* (the **range of X** is the set of possible values X can take on)

Probability Mass Function (PMF)

$P(X=k)$ **probability that the random variable X will take on the value k**

what is the probability of an outcome that will result in X being k

for discrete random variables (random variables with afinite, countably infinite range), this may sometimes be a piecewise function

Random Variables

☰ ❮ 01 02 03 04 ❯

Random variable

Captures a *quantitative property* *(some numerical value that describes the outcome)* of the outcome in a random experiment

e.g., sum of the dices on an random experiment where we roll 2 dice

Random Variables

01
02
03
04

An outcome from a random experiment → **X** a random variable → *Some number* (the **range (or support) of X** (sometimes denoted as Ω_X) is the set of possible values X can take on)

Probability Mass Function (PMF)

$$p_X(k) = P(X=k)$$

probability that the random variable X will take on the value k

what is the probability of an outcome that will result in X being k

for discrete random variables (random variables with a finite, countably infinite range), this may sometimes be a piecewise function

Random Variables

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Cumulative Distribution Function

$$F_X(k) =$$

P(X <= k) –> probability that the value X takes on is less than or equal to k

what is the probability of an outcome that will result in X being <= k

often can be derived from the PDF

Random Variables

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An outcome from a random experiment → **X** **a random variable** → *Some number* (the **range (or support) of X** (sometimes denoted as $\boldsymbol{\Omega_X}$) is the set of possible values X can take on)

Expectation

$$E[X] = \Sigma(k \cdot P(X=k))$$

sum of values in the range of X, weighted by the probability

on average, what value can we "expect" X to take?

think about it like a weighted average of all the possible values X could be (weighted by the P(X=k))

Random Variables

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Expectation

$$E[X] = \Sigma(k \cdot P(X=k))$$

sum of values in the range of X, weighted by the probability

on average, what value can we "expect" X to take?

just averaging all the possible values of X wouldn't work since each outcome isn't necessarily equally likely

Random Variables

01
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An outcome from a random experiment → **X** **a random variable** → *Some number*

(the **range (or support) of X** (sometimes denoted as $\mathbf{\Omega_X}$) is the set of possible values X can take on)

Expectation of a function of X (aka "Law of the Unconscious Statistician" (aka "LOTUS"))

$$E[f(X)] = \sum (f(k) \cdot P(X=k))$$

(note that the probabilities are still weighted using X (not f(X))

on average, what value can we "expect" f(X) to take?

LoE

Linearity **o**f **E**xpectation is a powerful property of random variables!

02 - Linearity of Expectation

Random Variables

allow us to represent a quantitative property of a random experiment

EXPECTATION - weighted average of possible outcomes

you could use "brute force" and use the formula for expectation ($E[X]=\sum(x*P(x))$)

sometimes, just applying the formula can be messy, so LoE comes in handy

LINEARITY OF EXPECTATION (LoE) *is one important property*

$$E(X+Y) = E(X) + E(Y)$$

the expected value of the sum of 2 random variables is the sum of their expected values

02 - Linearity of Expectation

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this gives us a helpful ***tool to calculate expectations of complex RVs***

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DECOMPOSE into a sum of random variables

$$X = X1 + X2 + \ldots + Xn$$

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APPLY linearity of expectation $E[X] = E[X1] + E[X2] + \ldots + E[Xn]$

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CONQUER and calculate each value $E[X1] = \ldots,\ E[X2] = \ldots,\ \ldots$

02 - Linearity of Expectation

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sometimes, these Xi variables we "decompose" X into are **indicator random variables**

DECOMPOSE into a sum of random variables — $X = X1 + X2 + ... + Xn$

APPLY linearity of expectation — $E[X] = E[X1] + E[X2] + ... + E[Xn]$

CONQUER and calculate each value — $E[X1] = ..., E[X2] = ..., ...$

02 - Linearity of Expectation

***X and Y DON'T have to be independent!

$$E(X+Y) = E(X) + E(Y)$$

the expected value of the sum of 2 random variables is the sum of their expected values

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*this gives us a helpful **tool to calculate expectations of complex RVs***

sometimes, these Xi variables we "decompose" X into are **indicator random variables**

DECOMPOSE into a sum of random variables	$X = X1 + X2 + \dots + Xn$
APPLY linearity of expectation	$E[X] = E[X1] + E[X2] + \dots + E[Xn]$
CONQUER and calculate each value	$E[X1] = \dots, E[X2] = \dots, \dots$

Indicator Random Variables

we can define a *indicator random variable* X for an event A

$$X = \begin{cases} 1 & \text{if event A happens} \\ 0 & \text{if event A doesn't happen} \end{cases}$$

^ X tells us whether event A will happen → so, P(X = 1) = P(A)

Note that **E[X]** = 1 * P(X=1) + 0 * P(X=0) **= P(X=1)**

this is why indicator RVs can be really useful when applying linearity of expectation!

Additional slides for content that will be covered later in the week!

02 - Linearity of Expectation

linearity of expectation is special!

E[X+Y] = E[X] + E[Y] *but* E[X^2] ≠ (E[X])^2

instead...

E[g(X)] = Σ(g(x) * P(X=x))

Variance

Variance is a another property of RVs (like expectation) that measures how much the values in the RV "vary"

03 - Variance

Random Variables

allow us to represent a quantitative property of a random experiment

VARIANCE - how "different" are values from the expectectation "on average"

every random variable has some variance

$$Var(X) = E[(X-E(X))^2] = \Sigma_x (P(X=x) * (x-E(X))^2)$$

expected value of the squared distance between each RV outcome and the expected value of RV

add up all the squared distances weighted by their probabilities

variance = (standard deviation)^2

03 - Variance

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Properties

$$Var(a \cdot X + b) = a^2 \cdot Var(X)$$

$$Var(X) = E[X^2] - (E[X])^2$$

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INDEPENDENT RV

What does independence mean for random variables?

Random Variable Independence

Random variables X and Y are **independent** if, *for all x, y in the ranges of X and Y*:

$$P(X=x, Y=y) = P(X=x) \cdot P(Y=y)$$

Knowing the value of X doesn't help "guess" what Y is

Random Variable Independence

Random variables X and Y are **independent** if –

$$P(X=x, Y=y) = P(X=x) \cdot P(Y=y)$$

Knowing the value of X doesn't help "guess" what Y is

it's a useful property! if X and Y are independent random variables then –

$$E(X \cdot Y) = E[X] \cdot E[Y]$$

$$Var(X + Y) = Var[X] + Var[Y]$$ *Linearity of variance holds*

Random Variable Independence

Random variables X and Y are **independent** if –

$$P(X=x, Y=y) = P(X=x) \cdot P(Y=y)$$

Knowing the value of X doesn't help "guess" what Y is

Additionally, there's **independent and identically distributed** (aka, "**i.i.d.** ") random variables

In addition to independence, i.i.d. random variables also **have the same pmf.**

For example, rolling a die twice, where X is the first roll number and Y is the second roll number

Problems!