Section 3

Review

- **Conditional Probability.** $\mathbb{P}(\mathcal{B} \mid \mathcal{A}) =$ ____________
- **Independent Events.** Two events $\mathcal{A}, \mathcal{B}$ are **independent** if $\mathbb{P}(\mathcal{A} \cap \mathcal{B}) =$ ____________.
  If $\mathbb{P}(\mathcal{A}) \neq 0$, this is equivalent to $\mathbb{P}(\mathcal{B} \mid \mathcal{A}) =$ ____________.
  If $\mathbb{P}(\mathcal{B}) \neq 0$, this is equivalent to $\mathbb{P}(\mathcal{A} \mid \mathcal{B}) =$ ____________.
- **Partition.** Nonempty events $\mathcal{E}_1, \ldots, \mathcal{E}_n$ partition the sample space Ω iff

  (1) ________________________ (2) ________________________

- **Bayes Rule.** For any events $\mathcal{A}$ and $\mathcal{B}$, $\mathbb{P}(\mathcal{A} \mid \mathcal{B}) =$ _______________ .
- **Chain Rule:** Suppose $\mathcal{A}_1, \ldots, \mathcal{A}_n$ are events. Then,

$$\mathbb{P}(\mathcal{A}_1 \cap \ldots \cap \mathcal{A}_n) = \underline{\hspace{3cm}}$$

- **Law of Total Probability (LTP)**: Suppose $\mathcal{E}_1, \ldots, \mathcal{E}_n$ is a partition of Ω and let $\mathcal{B}$ be any event. Then $\mathbb{P}(\mathcal{B}) = \sum_{i=1}^{n} \mathbb{P}(\mathcal{B} \cap \mathcal{E}_i) =$ ____________
- **Bayes Theorem with LTP:** $\mathcal{E}_1, \ldots, \mathcal{E}_n$ is a partition of Ω and let $\mathcal{B}$ be any event. Then

$$\mathbb{P}(\mathcal{E}_1 \mid \mathcal{B}) = \frac{\underline{\hspace{3cm}}}{\sum_{i=1}^{n} \underline{\hspace{3cm}}} .$$

The following will be covered in lecture on Friday.

- **Random Variable (rv)**: A numeric function $X : \Omega \to \mathbb{R}$ of the outcome.
- **Range/Support:** The support/range of a random variable X, denoted Ω_X, is the set of all possible values that X can take on.
- **Discrete Random Variable (drv)**: A random variable taking on a countable (either finite or countably infinite) number of possible values.
- **Probability Mass Function (pmf) for a discrete random variable $\mathbf{X}$**: a function $p_X : \Omega_X \to [0, 1]$ with $p_X(x) = \mathbb{P}(() X = x)$ that maps possible values of a discrete random variable to the probability of that value happening, such that $\sum_x p_X(x) = 1$.
- **Cumulative Distribution Function (CDF) for a random variable $\mathbf{X}$**: a function $F_X : R \to R$ with $F_X(x) = \mathbb{P}(() X \leqslant x)$
- **Expectation (expected value, mean, or average)**: The expectation of a discrete random variable is defined to be $\mathbb{E}[X] = \sum_x x p_X(x) = \sum_x x \mathbb{P}(() X = x)$. The expectation of a function of a discrete random variable $g(X)$ is $\mathbb{E}[g(X)] = \sum_x g(x) p_X(x)$.

Task 1 – Naive Bayes

This Ed Lesson (https://edstem.org/us/courses/50547/lessons/87272/slides/479664) is an introduction to using Bayes theorem to classify spam emails. This will be what you implement in the coding portion of PSet 3!

Task 2 – Flipping Coins

We consider two independent tosses of the same coin. The coin is "heads" one quarter of the time.

a) What is the probability that the second toss is "heads" given that the first toss is "tails"?

b) What is the probability that the second toss is "heads" given that at least one of the tosses is "tails"?

c) In the probability space of this task, give an example of two events that are disjoint but not independent.

d) In the probability space of this task, give an example of two events that are independent but not disjoint.

Task 3 – Balls from an Urn – Take 2

Say an urn contains three red balls and four blue balls. Imagine we draw three balls without replacement. (You can assume every ball is uniformly selected among those remaining in the urn.)

a) What is the probability that all three balls are all of the same color?

b) What is the probability that we get more than one red ball given the first ball is red?

Task 4 – Game Show

Corrupted by their power, the judges running the popular game show America's Next Top Mathematician have been taking bribes from many of the contestants. During each of two episodes, a given contestant is either allowed to stay on the show or is kicked off. If the contestant has been bribing the judges, they will be allowed to stay with probability 1. If the contestant has not been bribing the judges, they will be allowed to stay with probability $1/3$, independent of what happens in earlier episodes. Suppose that $1/4$ of the contestants have been bribing the judges. The same contestants bribe the judges in both rounds.

a) If you pick a random contestant, what is the probability that they are allowed to stay during the first episode?

b) If you pick a random contestant, what is the probability they are allowed to stay during both episodes?

c) If you pick a random contestant who was allowed to stay during the first episode, what is the probability that they get kicked off during the second episode?

d) If you pick a random contestant who was allowed to stay during the first episode, what is the probability that they were bribing the judges?

Task 5 – Allergy Season

In a certain population, everyone is equally susceptible to colds. Each person, in particular, catches a cold with probability 0.2.

The number of colds suffered by each person during each winter season ranges from 0 to 4, with probability 0.2 for each value (see table below). A new cold prevention drug is introduced that, for people for whom the drug is effective, changes the probabilities as shown in the table. Unfortunately, the effects of the drug last only the duration of one winter season, and the drug is only effective in 20% of people, independently.

number of colds	no drug or ineffective	drug effective
0	0.2	0.4
1	0.2	0.3
2	0.2	0.2
3	0.2	0.1
4	0.2	0.0

a) Sneezy decides to take the drug. Given that he gets 1 cold that winter, what is the probability that the drug is effective for Sneezy?

b) The next year he takes the drug again. Given that he gets 2 colds in this winter, what is the updated probability that the drug is effective for Sneezy?

c) Why is the answer to (b) the same as the answer to (a)?

Task 6 – Coins

There are three coins, C_1, C_2, and C_3. The probability of "heads" is 1 for C_1, 0 for C_2, and p for C_3. A coin is picked among these three uniformly at random, and then flipped a certain number of times.

a) What is the probability that the first n flips are tails?

b) Given that the first n flips were tails, what is the probability that C_1 was flipped? The probability that C_2 was flipped? The probability that C_3 was flipped?

Task 7 – Parallel Systems

A parallel system functions whenever at least one of its components works. Consider a parallel system of n components and suppose that each component works with probability p independently.

a) What is the probability the system is functioning?

b) If the system is functioning, what is the probability that component 1 is working?

c) If the system is functioning and component 2 is working, what is the probability that component 1 is working?

Task 8 – Marbles in Pockets

A girl has 5 blue and 3 white marbles in her left pocket, and 4 blue and 4 white marbles in her right pocket. If she transfers a randomly chosen marble from left pocket to right pocket without looking, and then draws a randomly chosen marble from her right pocket, what is the probability that it is blue?

Task 9 – A game

Pemi and Shreya are playing the following game: A 6-sided die is thrown and each time it's thrown, regardless of the history, it is equally likely to show any of the six numbers.

- If it shows 5, Pemi wins.
- If it shows 1, 2, or 6, Shreya wins.
- Otherwise, they play a second round and so on.

a) What is the probability that Shreya wins on the 4th round?

b) What is the probability that Shreya wins on the ith round?

c) What is the probability that Shreya wins? Use the fact that if $|x| < 1$, $\sum_{i=0}^{\infty} x^i = \frac{1}{1-x}$

Task 10 – Another game

Leiyi and Luxi are playing a tournament in which they stop as soon as one of them wins n games. Luxi wins each game with probability p and Leiyi wins with probability $1 - p$, independently of other games. What is the probability that Luxi wins and that when the match is over, Leiyi has won k games?

Task 11 – Balls from an Urn – Take 3

An urn contains 3 red and 3 blue balls with probability 3/5, 3 red and 1 blue balls with probability 1/10, and 5 red and 7 blue balls with probability 3/10. We draw a ball at random from the urn. Let R be the event that we draw a red ball. Let $xRyB$ be the event that the urn contains x red balls and y blue balls.
Are the events R and $3R3B$ independent?

Task 12 – Random Variables

(The material for this problem will be covered on Friday.)

Assume that we roll a fair 3-sided die three times. Here, the sides have values $1, 2, 3$.

a) Describe the PMF of the random variable X giving the sum of the first two rolls.

b) Give the expectation $\mathbb{E}[X]$.

c) Compute $\mathbb{P}(X > 3)$.

d) Let Y be the random variable describing the sum of the three rolls. Compute $\mathbb{P}(X = 5 \mid Y = 7)$.

Task 13 – Hungry Washing Machine

You have 10 pairs of socks (so 20 socks in total), with each pair being a different color. You put them in the washing machine, but the washing machine eats 4 of the socks chosen at random. Every subset of 4 socks is equally probable to be the subset that gets eaten. Let X be the number of complete pairs of socks that you have left.

a) What is the range of X, Ω_X (the set of possible values it can take on)? What is the probability mass function of X?

b) Find $\mathbb{E}[X]$ from the definition of expectation.