Section 2 – Solutions

Review

- **Binomial Theorem:** $\forall x, y \in, \forall n \in$: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$
- **Principle of Inclusion-Exclusion (PIE):** 2 events: $|A \cup B| = |A| + |B| - |A \cap B|$
  3 events: $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$
  In general: +singles - doubles + triples - quads + ...
- **Stars and Bars:** There are $\binom{n+k-1}{n} = \binom{n+k-1}{k-1}$ ways to pick n objects from k groups (where order doesn't matter and every element of each group is indistinguishable).
- **Pigeonhole Principle:** If there are n pigeons with k holes and $n > k$, then at least one hole contains at least 2 (or to be precise, nk) pigeons.
- **Complementary Counting (Complementing):** If asked to find the number of ways to do X, you can: (1) find the total number of ways to do everything and then (2) subtract the number of ways to *not* do X.
- **Sample Space:** The set of all possible outcomes of an experiment, denoted Ω or S
- **Event:** Some subset of the sample space, usually a capital letter such as $E \subseteq \Omega$
- **Union:** The union of two events E and F is denoted $E \cup F$
- **Intersection:** The intersection of two events E and F is denoted $E \cap F$ or EF
- **Mutually Exclusive:** Events E and F are mutually exclusive iff $E \cap F = \varnothing$
- **Complement:** The complement of an event E is denoted E^C or $\overline{E}$ or $\neg E$, and is equal to $\Omega \backslash E$
- **DeMorgan's Laws:** $(E \cup F)^C = E^C \cap F^C$ and $(E \cap F)^C = E^C \cup F^C$
- **Probability of an event E:** denoted $\mathbb{P}(E)$ or $\mathbb{P}(E)$ or $P(E)$

Axioms of Probability and their Consequences

1. **(Non-negativity)** For any event E, $\mathbb{P}(E) \geqslant 0$
2. **(Normalization)** $\mathbb{P}(\Omega) = 1$
3. **(Additivity)** If E and F are mutually exclusive, then $\mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F)$

Corollaries of these axioms:

- $\mathbb{P}(E) + \mathbb{P}(E^C) = 1$
- If $E \subseteq F$, $\mathbb{P}(E) \leqslant \mathbb{P}(F)$
- $\mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F) - \mathbb{P}(E \cap F)$

Equally Likely Outcomes: If every outcome in a finite sample space Ω is equally likely, and E is an event, then $\mathbb{P}(E) = \dfrac{|E|}{|\Omega|}$.

- Make sure to be consistent when counting $|E|$ and $|\Omega|$. Either order matters in both, or order doesn't matter in both.

Task 1 – Review Questions

a) **True** or **False**. The following statement is always true: $|A \cup B| = |A| + |B|$

False. Unless A and B do not overlap, by the property of inclusion-exclusion, $|A \cup B| = |A| + |B| - |A \cap B|$.

b) If there are 7 pigeons that each go into one of 3 holes:

◯ There is at least one hole with exactly 3 pigeons in it.
◯ There is at least one hole with at least 3 pigeons in it.
◯ There is exactly one hole with at least 3 pigeons in it.

By the pigeonhole principle, there will be *at least* one hole with *at least* $\lceil \frac{7}{3} \rceil = 3$ pigeons in it.

c) $(x + y)^n =$

◯ $\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$
◯ $\sum_{k=0}^{n} x^k y^{n-k}$
◯ $\sum_{k=0}^{n} \binom{n}{k} x^k$

(a) by definition of binomial theorem.

d) An **event** and **sample space** are, respectively:

◯ The total set of possible outcomes; A subset of the event space
◯ A subset of the sample space; The total set of possible outcomes
◯ Some set of outcomes; Any other set of outcomes.

(b) by definition of event and sample space. An event is always a subset of the sample space.

e) **True** or **False**. It is always true that $\mathbb{P}(E) = \frac{|E|}{|\Omega|}$.

False. This is only true if all outcomes in the sample space are *equally likely*.

f) If A is the event that I eat an apple today,

◯ $\overline{A}$ is the event that I eat a banana today, and $P(A) + P(\overline{A}) = 0.5$
◯ $\overline{A}$ is the event that I do not eat an apple today, and $P(A) + P(\overline{A}) = 0$
◯ $\overline{A}$ is the event that I do not eat an apple today, and $P(A) + P(\overline{A}) = 1$

(c) is correct. $\overline{A}$ is the *complement* of the event A. In this case, that is the event that I do *not* eat an apple today. $P(A) + P(\overline{A}) = 1$ since I will definitely either eat or not eat an apple. This property holds for all events!

g) True or **False**. For any two events A and B $P(A \cup B) > P(A) + P(B)$.

False. The correct statement would be $P(A \cup B) \geqslant P(A) + P(B)$. If A and B are mutually exclusive (i.e., $A \cap B$ is an empty set), then $P(A \cup B) = P(A) + P(B)$. Another way to think about it is, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ (this is similar to the property of inclusion-exclusion!). Sometimes, $P(A \cap B)$ might be 0 when A and B can't happen at the same time, so then $P(A \cup B)$ would equal $P(A) + P(B)$

Task 2 – HBCDEFGA

How many ways are there to permute the 8 letters A, B, C, D, E, F, G, H so that A is not at the beginning and H is not at the end?

The total number of permutations is $8!$. The number of permutations with A at the beginning is $7!$ and the number with H at the end is $7!$. By inclusion/exclusion, the number that have either A at the beginning or H at the end or both is $2 \cdot 7! - 6!$ since there are $6!$ that have A at the beginning and H at the end. Finally, using complementary counting, the number that have neither A at the end or H at the end is $8! - (2 \cdot 7! - 6!)$.

Task 3 – Ingredients

Find the number of ways to rearrange the word "INGREDIENT", such that no two identical letters are adjacent to each other. For example, "INGREEDINT" is invalid because the two E's are adjacent. Hint: use inclusion-exclusion.

We use inclusion-exclusion. Let Ω be the set of all anagrams (permutations) of "INGREDIENT", and A_I be the set of all anagrams with two consecutive I's. Define A_E and A_N similarly. $A_I \cup A_E \cup A_N$ clearly are the set of anagrams we don't want. So we use complementing to count the size of $\Omega \backslash (A_I \cup A_E \cup A_N)$. By inclusion exclusion, $|A_I \cup A_E \cup A_N|$ =singles-doubles+triples, and by complementing, $|\Omega \backslash (A_I \cup A_E \cup A_N)| = |\Omega| - |A_I \cup A_E \cup A_N|$.

First, $|\Omega| = \frac{10!}{2!2!2!}$ because there are 2 of each of I,E,N's (multinomial coefficient). Clearly, the size of A_I is the same as A_E and A_N. So $|A_I| = \frac{9!}{2!2!}$ because we treat the two adjacent I's as one entity. We also need $|A_I \cap A_E| = \frac{8!}{2!}$ because we treat the two adjacent I's as one entity and the two adjacent E's as one entity (same for all doubles). Finally, $|A_I \cap A_E \cap A_N| = 7!$ since we treat each pair of adjacent I's, E's, and N's as one entity.

Putting this together gives $\boxed{\frac{10!}{2!2!2!} - \left(\binom{3}{1} \cdot \frac{9!}{2!2!} - \binom{3}{2} \cdot \frac{8!}{2!} + \binom{3}{3} \cdot 7! \right)}$

Task 4 – Count the Solutions

Consider the following equation: $a_1 + a_2 + a_3 + a_4 + a_5 + a_6 = 70$. A solution to this equation over the nonnegative integers is a choice of a nonnegative integer for each of the variables $a_1, a_2, a_3, a_4, a_5, a_6$ that satisfies the equation. For example, $a_1 = 15, a_2 = 3, a_3 = 15, a_4 = 0, a_5 = 7, a_6 = 30$ is a solution. To be different, two solutions have to differ on the value assigned to some a_i. How many different solutions are there to the equation?

(Hint: Think about splitting a sequence of 70 1's into 6 blocks, each block consisting of consecutive 1's in the sequence. The number of 1's in the i-th block corresponds to the value of a_i. Note that the i-th block is allowed to be empty, corresponding to $a_i = 0$.)

Using the stars and bars method, we get:

$$\binom{70+6-1}{6-1} = \binom{75}{5} = 17,259,390$$

Task 5 – Card Party

At a card party, someone brings out a deck of bridge cards (4 suits with 13 cards in each). N people each pick 2 cards from the deck and hold onto them. What is the minimum value of N that guarantees at least 2 people have the same combination of suits?

$\boxed{N = 11}$: There are $\binom{4}{2}$ combinations of 2 different suits, plus 4 possibilities of having 2 cards of the same suit. This gives 10 different possible combinations of suits (the pigeonhole). Since each person (pigeon) will fall into 1 pigeonhole (they will end up with some combination of suits), with $N = 11$ you can apply the pigeonhole principle. That's because with $N = 11$, there are 11 pigeons and 10 pigeonholes, so at least one pigeonhole will have at least 2 pigeons - 2 people will have the same combination of suits.

Task 6 – The Pigeonhole Principle

Show that in any group of n people there are two who have an identical number of friends within the group. (Friendship is bi-directional – i.e., if A is friend of B, then B is friend of A – and nobody is a friend of themselves.)

Solve in particular the following two cases individually:

a) Everyone has at least one friend.

Everyone has between 1 and $n-1$ friends (i.e., $n-1$ holes), and there are n people (the "pigeons"). Since $n > n-1$, we can apply the pigeonhole principle to say that at least two people (pigeons) will have the same number of friends (will fall into the same pigeonhole).

b) At least one person has no friends.

Here, we need to observe that if someone has 0 friends, then nobody has $n-1$ friends (by the symmetry of the friendship relation). Then, possible choices are now between 0 and $n-2$ friends (i.e., $n-1$ holes), and there are n people (the "pigeons"). Therefore, applying the pigeonhole principle since $n > n-1$, two of them will have the same number of friends (two pigeons, people, will fall into the same pigeonhole, number of friends).

Task 7 – A Team and a Captain

Give a combinatorial proof of the following identity:

$$n\binom{n-1}{r-1} = \binom{n}{r}r.$$

Hint: Consider two ways to choose a team of size r out of a set of size n and a captain of the team (who is also one of the team members).

Remember that a combinatorial proof just requires that we show both sides are equivalent ways of counting a situation.

Left hand side: Choose a team of size r and a captain for that team (from among the r) by first choosing the captain (n choices) and then choosing the rest of the team $\binom{n-1}{r-1}$.

Right hand side: Choose a team of size r and a captain for that team by first choosing the team ($\binom{n}{r}$ choices) and then choosing the captain from among the members of the team (r choices).

Task 8 – Balls from an Urn

Say an urn (a fancy name for a jar that doesn't have a lid) contains one red ball, one blue ball, and one green ball. (Other than for their colors, balls are identical.) Imagine we draw two balls *with replacement*, i.e., after drawing one ball, with put it back into the urn, before we draw the second one. (In particular, each ball is equally likely to be drawn.)

a) Give a probability space describing the experiment.

$\Omega = \{B, R, G\}^2$ and $\mathbb{P}(\omega) = 1/9$ for all $\omega \in \Omega$.

b) What is the probability that both balls are red? (Describe the event first, before you compute its probability.)

The event is $\mathcal{A} = \{RR\}$. Its probability is $\mathbb{P}(\mathcal{A}) = \frac{|\mathcal{A}|}{9} = \frac{1}{9}$ because there are $3^2 = 9$ outcomes in the sample space, and all outcomes are equally likely.

c) What is the probability that at most one ball is red?

This is just $\mathcal{A}^c$, the complement of $\mathcal{A}$. We know that $\mathbb{P}(\mathcal{A}^c) = 1 - \mathbb{P}(\mathcal{A}) = 1 - \frac{1}{9} = 8/9$.

d) What is the probability that we get at least one green ball?

This is the event $\mathcal{B} = \{GR, GB, GG, RG, BG\}$, and thus $\mathbb{P}(\mathcal{B}) = \frac{|\mathcal{B}|}{9} = \frac{5}{9}$.

e) Repeat **b)-d)** for the case where the balls are drawn *without replacement*, i.e., when the first ball is drawn, it is not placed back from the urn. Thus the two balls drawn have different colors. (Note that this will still be a uniform probability space.

Here, the probability space changes: First of all, the outcomes RR, GG, BB are not possible any more, so let us remove them from Ω, which is now $\Omega = \{BG, BR, GB, GR, RB, RG\}$. As before each outcome is equally likely and has probability $1/6$. One way to think about this is that $\mathbb{P}(\omega) = \frac{1}{2\cdot 3} = \frac{1}{6}$ for every outcome because we have three choices for the first ball, but only two for the second.

It can never be that both balls are red – therefore, for **b)**, the probability becomes 0. For **c)**, the probability is 1, and for **d)**, the event becomes $\mathcal{B} = \{GR, GB, RG, BG\}$, and $\mathbb{P}(\mathcal{B}) = 4 \cdot \frac{1}{6} = \frac{2}{3}$.

Task 9 – Spades and Hearts

Given 3 different spades and 3 different hearts, shuffle them. (i) What is the sample space and how big is it? (ii) What is the probability of each outcome in the sample space? (iii) What is $\mathbb{P}(E)$, where E is the event that the suits of the shuffled cards are in alternating order?

The sample space Ω is all re-orderings possible: there are $|\Omega| = 6!$ such. Each outcome represents one ordering and all of them have equal probability $\frac{1}{6!}$. Now for E, order the spades and hearts independently, so there are $3!^2$ ways to do so. Finally choose whether you want hearts or spades first. All such orderings are equally likely, so $\mathbb{P}(E) = \frac{|E|}{|\Omega|} = \frac{2\cdot 3!^2}{6!}$.

Task 10 – Congressional Tea

Twenty politicians are having tea, 6 Democrats and 14 Republicans.

a) If they only give tea to 10 of the 20 people, what is the probability that they only give tea to Republicans? (We assume every possible way of giving tea is equally likely.)

> The sample space is all possible ways of choosing which 10 people get tea, so $|\Omega| = \binom{20}{10}$ ways. The event is the ways to give tea to only Republicans, of which there are $\binom{14}{10}$ ways. So the probability is $\frac{\binom{14}{10}}{\binom{20}{10}}$.

b) If they only give tea to 10 of the 20 people, what is the probability that they give tea to 8 Republicans and 2 Democrats? (We assume every possible way of giving tea is equally likely.)

> Similarly to the previous part, $\frac{\binom{14}{8}\binom{6}{2}}{\binom{20}{10}}$.

Task 11 – Shuffling Cards

We have a deck of cards, with 4 suits, and 13 cards in each suit. Within each suit, the cards are ordered Ace > King > Queen > Jack > 10 > $\cdots$ > 2. Also, suppose we perfectly shuffle the deck (i.e., all possible shuffles are equally likely).

What is the probability the first card on the deck is (strictly) larger than the second one?

First off, the sample space Ω here consists of all pairs of cards – which we can represent by their value *and* suit, e.g., $(4\clubsuit, A\diamondsuit)$. There $52 \cdot 51$ possible outcomes, therefore $\mathbb{P}(\omega) = \frac{1}{52 \cdot 51}$ for all $\omega \in \Omega$.

Let us now look at the size of the event $\mathcal{E}$ containing all pairs where the first card is strictly larger than the second. Then, the number of pairs of values of cards a and b where $a < b$ is exactly $\binom{13}{2}$. We can then assign suits to each of them – given the cards are different, all suits are possible for each, so there are $4^2 = 16$ choices. Thus, overall,

$$|\mathcal{E}| = \binom{13}{2} \cdot 16 \,.$$

Therefore,

$$\mathbb{P}(\mathcal{E}) = \frac{|\mathcal{E}|}{|\Omega|} = \frac{\binom{13}{2} \cdot 16}{52 \cdot 51} = \frac{8}{17} \,.$$

Task 12 – Robot Wears Socks

Suppose Joe is a k-legged robot, who wears a sock and a shoe on each leg. Suppose he puts on k socks and k shoes in some order, each equally likely. Each action is specified by saying whether he puts on a sock or a shoe, and saying which leg he puts it on. In how many ways can he put on his socks and shoes in a valid order? We say an ordering is valid if, for every leg, the sock gets put on before the shoe. Assume all socks are indistinguishable from each other, and all shoes are indistinguishable from each other.

First, note that there are $2k$ possible actions which we will denote by $Sock_1, Shoe_1, \ldots, Sock_k, Shoe_k$. Here $Sock_i$ means that a sock is placed on leg i and similarly $Shoe_j$ means that a shoe is placed on leg j.

One way to approach the problem is by imagining that we have $2k$ empty slots and that each action can be placed in exactly one of the slots. We can denote the set of slots by $\{1, 2, \ldots, 2k\}$. First, we assign the pair of actions $Sock_1$ and $Shoe_1$ to two of the $2k$ slots. Note that the two actions must be ordered $Sock_1, Shoe_1$ in a valid ordering. Thus, any choice of two positions(i.e. a subset of size two from $\{1, 2, \ldots, 2k\}$) will correspond to exactly one valid ordering of the two actions. Hence, we have $\binom{2k}{2}$ valid assignments for the pair of actions $Sock_1, Shoe_1$. Next, we assign the pair of actions $Sock_2$ and $Shoe_2$ to two of the remaining $2k-2$ slots. By the same reasoning, there are $\binom{2k-2}{2}$ valid assignments. We assign actions until we arrive at $Sock_k$ and $Shoe_k$ for which we only have two slots left and thus $\binom{2}{2} = 1$ valid assignments. Using the product rule, we have

$$\binom{2k}{2}\binom{2k-2}{2}\cdots\binom{2}{2} = \frac{2k!}{2^k}$$

total possible actions.

Alternatively, suppose we describe a sequences of actions such as $Sock_1, Shoe_1, Sock_2, Shoe_2, \ldots, Sock_k, Shoe_k$. There are $(2k)!$ ways to order these actions. Clearly many of these orderings are not valid since it will often be the case that we put on a shoe before a sock on at least one of the legs. So, let us eliminate these invalid orderings by focusing on one leg at a time. First, we will focus on the pair of actions corresponding to the first leg, $Sock_1, Shoe_1$. We can eliminate half of the $(2k)!$ orderings because half of them will have $Shoe_1$ placed before $Sock_1$. Thus, we have $(2k)!/2$ orderings remaining, orderings in which $Sock_1, Shoe_1$ are in correct order, while the remaining actions might not. Next, we move on to the second leg and the pair of actions $Sock_2, Shoe_2$. Again, we can eliminate half of the $(2k)!/2$ remaining orderings where $Shoe_2$ is placed before $Sock_2$. This leaves us with $(2k)!/2^2$ orderings in which the pairs $Sock_1, Shoe_1$ and $Sock_2, Shoe_2$ are placed correctly, but the remaining actions might not. Repeating the process for all the k legs, we obtain $(2k)!/2^k$.

Task 13 – Trick or Treat

Suppose on Halloween, someone is too lazy to keep answering the door, and leaves a jar of exactly N total candies. You count that there are exactly K of them which are kit kats (and the rest are not). The sign says to please take exactly n candies. Each subset of size n is equally likely to be drawn (and they are drawn all at once, so order doesn't matter). Let E be the event that you draw exactly k kit kats. What is $\mathbb{P}(E)$?

The sample space consists of all ways of choosing a subset of n candies out of a total of N, so $|\Omega| = \binom{N}{n}$. Therefore,

$$\boxed{\mathbb{P}(E) = \frac{|E|}{|\Omega|} = \frac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}}}$$

since an outcome in E can be chosen by first selecting which subset of k of the K kitkats are drawn and then which subset of $n-k$ of the $N-K$ other candies are drawn.

Task 14 – Weighted Die

Consider a weighted die such that

- $\mathbb{P}(1) = \mathbb{P}(2)$,
- $\mathbb{P}(3) = \mathbb{P}(4) = \mathbb{P}(5) = \mathbb{P}(6)$, and
- $\mathbb{P}(1) = 3\mathbb{P}(3)$.

What is the probability that the outcome is 3 or 4?

By the second axiom of probability, the sum of probabilities for the sample space must equal 1. That is, $\sum_{i=1}^{6} \mathbb{P}(i) = 1$. Since $\mathbb{P}(1) = \mathbb{P}(2)$ and $\mathbb{P}(1) = 3\mathbb{P}(3)$, we have that: $1 = \mathbb{P}(1) + \mathbb{P}(2) + \mathbb{P}(3) + \mathbb{P}(4) + \mathbb{P}(5) + \mathbb{P}(6) = 3\mathbb{P}(3) + 3\mathbb{P}(3) + \mathbb{P}(3) + \mathbb{P}(3) + \mathbb{P}(3) + \mathbb{P}(3) = 10\mathbb{P}(3)$

Thus, solving algebraically, $\mathbb{P}(3) = 0.1$, so $\mathbb{P}(3) = \mathbb{P}(4) = 0.1$. Since rolling a 3 and 4 are disjoint events, then $\mathbb{P}(3 \text{ or } 4) = \mathbb{P}(3) + \mathbb{P}(4) = 0.1 + 0.1 = 0.2$.