Section 2

Review

- **Binomial Theorem:** $\forall x, y \in, \forall n \in$: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$
- **Principle of Inclusion-Exclusion (PIE):** 2 events: $|A \cup B| = |A| + |B| - |A \cap B|$
  3 events: $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$
  In general: +singles - doubles + triples - quads + ...
- **Stars and Bars:** There are $\binom{n+k-1}{n} = \binom{n+k-1}{k-1}$ ways to pick n objects from k groups (where order doesn't matter and every element of each group is indistinguishable).
- **Pigeonhole Principle:** If there are n pigeons with k holes and $n > k$, then at least one hole contains at least 2 (or to be precise, nk) pigeons.
- **Complementary Counting (Complementing):** If asked to find the number of ways to do X, you can: (1) find the total number of ways to do everything and then (2) subtract the number of ways to *not* do X.
- **Sample Space:** The set of all possible outcomes of an experiment, denoted Ω or S
- **Event:** Some subset of the sample space, usually a capital letter such as $E \subseteq \Omega$
- **Union:** The union of two events E and F is denoted $E \cup F$
- **Intersection:** The intersection of two events E and F is denoted $E \cap F$ or EF
- **Mutually Exclusive:** Events E and F are mutually exclusive iff $E \cap F = \varnothing$
- **Complement:** The complement of an event E is denoted E^C or $\overline{E}$ or $\neg E$, and is equal to $\Omega \backslash E$
- **DeMorgan's Laws:** $(E \cup F)^C = E^C \cap F^C$ and $(E \cap F)^C = E^C \cup F^C$
- **Probability of an event E:** denoted $\mathbb{P}(E)$ or $\mathbb{P}(E)$ or $P(E)$

Axioms of Probability and their Consequences

1. **(Non-negativity)** For any event E, $\mathbb{P}(E) \geqslant 0$
2. **(Normalization)** $\mathbb{P}(\Omega) = 1$
3. **(Additivity)** If E and F are mutually exclusive, then $\mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F)$

Corollaries of these axioms:

- $\mathbb{P}(E) + \mathbb{P}\left(E^C\right) = 1$
- If $E \subseteq F$, $\mathbb{P}(E) \leqslant \mathbb{P}(F)$
- $\mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F) - \mathbb{P}(E \cap F)$

Equally Likely Outcomes: If every outcome in a finite sample space Ω is equally likely, and E is an event, then $\mathbb{P}(E) = \frac{|E|}{|\Omega|}$.

- Make sure to be consistent when counting $|E|$ and $|\Omega|$. Either order matters in both, or order doesn't matter in both.

Task 1 – Review Questions

a) **True** or **False**. The following statement is always true: $|A \cup B| = |A| + |B|$

b) If there are 7 pigeons that each go into one of 3 holes:

- ◯ There is at least one hole with exactly 3 pigeons in it.
- ◯ There is at least one hole with at least 3 pigeons in it.
- ◯ There is exactly one hole with at least 3 pigeons in it.

c) $(x + y)^n =$

- ◯ $\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$
- ◯ $\sum_{k=0}^{n} x^k y^{n-k}$
- ◯ $\sum_{k=0}^{n} \binom{n}{k} x^k$

d) An **event** and **sample space** are, respectively:

- ◯ The total set of possible outcomes; A subset of the event space
- ◯ A subset of the sample space; The total set of possible outcomes
- ◯ Some set of outcomes; Any other set of outcomes.

e) **True** or **False**. It is always true that $\mathbb{P}(E) = \frac{|E|}{|\Omega|}$.

f) If A is the event that I eat an apple today,

- ◯ $\overline{A}$ is the event that I eat a banana today, and $P(A) + P(\overline{A}) = 0.5$
- ◯ $\overline{A}$ is the event that I do not eat an apple today, and $P(A) + P(\overline{A}) = 0$
- ◯ $\overline{A}$ is the event that I do not eat an apple today, and $P(A) + P(\overline{A}) = 1$

g) **True** or **False**. For any two events A and B $P(A \cup B) > P(A) + P(B)$.

Task 2 – HBCDEFGA

How many ways are there to permute the 8 letters A, B, C, D, E, F, G, H so that A is not at the beginning and H is not at the end?

Task 3 – Ingredients

Find the number of ways to rearrange the word "INGREDIENT", such that no two identical letters are adjacent to each other. For example, "INGREEDINT" is invalid because the two E's are adjacent. Hint: use inclusion-exclusion.

Task 4 – Count the Solutions

Consider the following equation: $a_1 + a_2 + a_3 + a_4 + a_5 + a_6 = 70$. A solution to this equation over the nonnegative integers is a choice of a nonnegative integer for each of the variables $a_1, a_2, a_3, a_4, a_5, a_6$ that satisfies the equation. For example, $a_1 = 15, a_2 = 3, a_3 = 15, a_4 = 0, a_5 = 7, a_6 = 30$ is a solution. To be different, two solutions have to differ on the value assigned to some a_i. How many different solutions are there to the equation?

Task 5 – Card Party

At a card party, someone brings out a deck of bridge cards (4 suits with 13 cards in each). N people each pick 2 cards from the deck and hold onto them. What is the minimum value of N that guarantees at least 2 people have the same combination of suits?

Task 6 – The Pigeonhole Principle

Show that in any group of n people there are two who have an identical number of friends within the group. (Friendship is bi-directional – i.e., if A is friend of B, then B is friend of A – and nobody is a friend of themselves.)

Solve in particular the following two cases individually:

a) Everyone has at least one friend.

b) At least one person has no friends.

Task 7 – A Team and a Captain

Give a combinatorial proof of the following identity:

$$n\binom{n-1}{r-1} = \binom{n}{r}r.$$

Hint: Consider two ways to choose a team of size r out of a set of size n and a captain of the team (who is also one of the team members).

Task 8 – Balls from an Urn

Say an urn (a fancy name for a jar that doesn't have a lid) contains one red ball, one blue ball, and one green ball. (Other than for their colors, balls are identical.) Imagine we draw two balls *with replacement*, i.e., after drawing one ball, with put it back into the urn, before we draw the second one. (In particular, each ball is equally likely to be drawn.)

a) Give a probability space describing the experiment.

b) What is the probability that both balls are red? (Describe the event first, before you compute its probability.)

c) What is the probability that at most one ball is red?

d) What is the probability that we get at least one green ball?

e) Repeat **b)-d)** for the case where the balls are drawn *without replacement*, i.e., when the first ball is drawn, it is not placed back from the urn. Thus the two balls drawn have different colors. (Note that this will still be a uniform probability space.

Task 9 – Spades and Hearts

Given 3 different spades and 3 different hearts, shuffle them. (i) What is the sample space and how big is it? (ii) What is the probability of each outcome in the sample space? (iii) What is $\mathbb{P}(E)$, where E is the event that the suits of the shuffled cards are in alternating order?

Task 10 – Congressional Tea

Twenty politicians are having tea, 6 Democrats and 14 Republicans.

a) If they only give tea to 10 of the 20 people, what is the probability that they only give tea to Republicans? (We assume every possible way of giving tea is equally likely.)

b) If they only give tea to 10 of the 20 people, what is the probability that they give tea to 8 Republicans and 2 Democrats? (We assume every possible way of giving tea is equally likely.)

Task 11 – Shuffling Cards

We have a deck of cards, with 4 suits, and 13 cards in each suit. Within each suit, the cards are ordered Ace > King > Queen > Jack > 10 > $\cdots$ > 2. Also, suppose we perfectly shuffle the deck (i.e., all possible shuffles are equally likely).

What is the probability the first card on the deck is (strictly) larger than the second one?

Task 12 – Robot Wears Socks

Suppose Joe is a k-legged robot, who wears a sock and a shoe on each leg. Suppose he puts on k socks and k shoes in some order, each equally likely. Each action is specified by saying whether he puts on a sock or a shoe, and saying which leg he puts it on. In how many ways can he put on his socks and shoes in a valid order? We say an ordering is valid if, for every leg, the sock gets put on before the shoe. Assume all socks are indistinguishable from each other, and all shoes are indistinguishable from each other.

Task 13 – Trick or Treat

Suppose on Halloween, someone is too lazy to keep answering the door, and leaves a jar of exactly N total candies. You count that there are exactly K of them which are kit kats (and the rest are not). The sign says to please take exactly n candies. Each subset of size n is equally likely to be drawn (and they are drawn all at once, so order doesn't matter). Let E be the event that you draw exactly k kit kats. What is $\mathbb{P}(E)$?

Task 14 – Weighted Die

Consider a weighted die such that

- $\mathbb{P}(1) = \mathbb{P}(2)$,
- $\mathbb{P}(3) = \mathbb{P}(4) = \mathbb{P}(5) = \mathbb{P}(6)$, and
- $\mathbb{P}(1) = 3\mathbb{P}(3)$.

What is the probability that the outcome is 3 or 4?