# Section 1 – Solutions

## Review

- **Sum rule.** If you can choose from EITHER one of $n$ options, OR one of $m$ options with NO overlap with the previous $n$, then the number of possible outcomes of the experiment is $n + m$

- **Product rule.** In a sequential process with $m$ steps, if there are $n_1$ choices for the 1st step, $n_2$ choices for the 2nd step (given the first choice), ..., and $n_m$ choices for the $m$th step (given the previous choices), then the total number of outcomes is $n_1 n_2 ... n_m$

- **Number of ways to order $n$ distinct objects:** $n! = n \cdot (n-1) \cdots 3 \cdot 2 \cdot 1$

- **Number of ways to select from $n$ distinct objects:**
  - **Permutations** (number of ways to linearly arrange $k$ objects out of $n$ distinct objects, when the order of the $k$ objects matters):
  $$P(n,k) = \frac{n!}{(n-k)!}$$
  - **Combinations** (number of ways to choose $k$ objects out of $n$ distinct objects, when the order of the $k$ objects does not matter):
  $$\frac{n!}{k!(n-k)!} = \binom{n}{k} = C(n,k)$$

- **Complementary Counting:** If asked to find the number of ways to do $X$, you can: (1) find the total number of ways to do everything and then (2) subtract the number of ways to *not* do $X$.

  *The rest of these will be covered in class on Friday.*

- **Binomial theorem.** $\forall x, y \in \mathbb{R}, \forall n \in \mathbb{N}$: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$

- **Principle of Inclusion-Exclusion (PIE):** 2 events: $|A \cup B| = |A| + |B| - |A \cap B|$
  3 events: $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$
  In general: +singles - doubles + triples - quads + ...

- **Counting when there are repeats - multinomial coefficients.** Suppose there are $n$ objects, but only $k$ are distinct, with $k \leqslant n$. (For example, "godoggy" has $n = 7$ objects (characters) but only $k = 4$ are distinct: $(g, o, d, y)$). Let $n_i$ be the number of times object $i$ appears, for $i \in \{1, 2, \ldots, k\}$. (For example, $(3, 2, 1, 1)$, continuing the "godoggy" example.) The number of distinct ways to arrange the $n$ objects is $\binom{n}{n_1, \ldots, n_k} = \frac{n!}{n_1! \cdot \ldots \cdot n_k!}$

## Task 1 – Sets

(a) For each one of the following sets, give its **cardinality**, i.e., indicate how many elements it contains:

- $A = \varnothing$ - $B = \{\varnothing\}$ - $C = \{\{\varnothing\}\}$ - $D = \{\varnothing, \{\varnothing\}\}$

- $|A| = 0$ - $|B| = 1$ - $|C| = 1$ - $|D| = 2$

(b) Let $S = \{a, b, c\}$ and $T = \{c, d\}$. Compute:

- $S \cup T$ - $S \cap T$ - $S \backslash T$ - $2^{S \backslash T}$ - $S \times T$

- $S \cup T = \{a, b, c, d\}$ because those elements are in at least one of the sets
- $S \cap T = \{c\}$ because that's the only element that is in both sets.
- $S \backslash T = \{a, b\}$ because those are the elements in S that are not in T.
- $2^{S \backslash T} = \{\varnothing, \{a\}, \{b\}, \{a, b\}\}$ because that is the power set (all subsets of the set) of the set found in the previous set.
- $S \times T = \{(a, c), (a, d), (b, c), (b, d), (c, c), (c, d)\}$ because that is the set of all ordered pairs from those two sets.

## Task 2 – Basic Counting

**a)** Credit-card numbers are made of 15 decimal digits, and a 16th checksum digit (which is uniquely determined by the first 15 digits). How many credit-card numbers are there?

Each of the 15 decimal digits have 10 options for what number it is. Then, the 16th digit has only 1 option after the first 15 digits are chosen, because it is uniquely chosen from the first 15 digits. So, by the product rule, there are $10^{15} \cdot 1 = 10^{15}$ options.

**b)** How many positive divisors does $1440 = 2^5 3^2 5$ have?

$6 \cdot 3 \cdot 2 = 36$.

Every positive divisor of $1440$ can be written as $2^i 3^j 5^k$ where $i \in \{0, \ldots, 5\}$, $j \in \{0, 1, 2\}$, and $k \in \{0, 1\}$. Then, by the product rule, there are $6 \cdot 3 \cdot 2$ options since there are 6 possible values of i, 3 possible values of j, and 2 possible values of k.

**c)** How many ways are there to arrange the CSE 312 staff on a line (11 TAs, two professors) for a group picture?

$13!$. There are 13 people who we want to arrange in a line. Since order matters, we can use a permutation and find that there are $13!$ ways to order these 13 people

**d)** How many ways are there to arrange the CSE 312 staff on a line so that Professors Tessaro and Beame are at the two ends of the line?

$2 \cdot 11!$.

There are two options for which ends the professors are on and $11!$ ways to order the TAs. Then, by the product rule, there are $2 \cdot 11!$.

## Task 3 – Seating

How many ways are there to seat 10 people, consisting of 5 couples, in a row of 10 seats if ...

**a)** ... all couples are to get adjacent seats?

Consider each couple as a unit. Apply the product rule, first choosing one of the $5!$ permutations of the 5 couples, and then, for each couple in turn, choosing one of the 2 permutations for how they sit (for a total of $2^5$). Therefore, the answer is: $\boxed{5! \cdot 2^5.}$

**b)** ... anyone can sit anywhere, except that one couple insists on *not* sitting in adjacent seats?

Apply complementary counting to first compute the total number of arrangements of the 10 people, and then subtract from this the number of arrangements in which that particular couple does get adjacent seats. There are $10!$ for the former, and there are $9! \cdot 2$ arrangements in which this couple does sit in adjacent seats, since you can treat the couple as a unit, permute the 9 "individuals" (consisting of 8 people plus the couple) and then consider the 2 permutations for that couple. That means the answer to the question is $\boxed{10! - 9! \cdot 2 = 8 \cdot 9!}$.

Alternatively, we can do casework. Name the two people in the couple A and B. There are two cases: A can sit on one of the ends, or not. If A sits on an end seat, A has $2$ choices and B has $8$ possible seats. If A doesn't sit on the end, A has $8$ choices and B only has $7$. So there are a total of $2 \cdot 8 + 8 \cdot 7$ ways $A$ and $B$ can sit. Once they do, the other $8$ people can sit in $8!$ ways since there are no other restrictions. Hence the total number of ways is $\boxed{(2 \cdot 8 + 8 \cdot 7)8! = 9 \cdot 8 \cdot 8! = 8 \cdot 9!}$.

## Task 4 – Weird Card Game

In how many ways can a pack of fifty-two cards (in four suits of thirteen cards each) be dealt to thirteen players, four to each, so that every player has one card from each of the suits?

Apply the product rule: First deal the hearts, one to each person, then the spades, one to each person, then diamonds, then the clubs. For each of these steps, there are $13!$ possibilities. Therefore, the answer is $\boxed{13!^4}$.

## Task 5 – Full Class

There are 40 seats and 40 students in a classroom. Suppose that the front row contains 10 seats, and there are 5 students who must sit in the front row in order to see the board clearly. How many seating arrangements are possible with this restriction?

Use the product rule: First choose the seats that the five students who must sit in the front row will take and then choose their placement in these seats. For this, there are $\binom{10}{5} \cdot 5!$ choices. Then arrange the remaining 35 students in the remaining 35 seats ($35!$ ways).

So the answer is $\boxed{\binom{10}{5} \cdot 5! \cdot 35!}$.

## Task 6 – Escape the Professor

There are 6 security professors and 7 theory professors taking part in an escape room. The solution requires that they choose 4 pairs, each consisting of one security professor and one theory professor. How many options for pairings do they have?

> $\boxed{\binom{6}{4}\binom{7}{4}4!}$. Apply the product rule to first choose $4$ of the security professors, then $4$ of the theory professors. Then assign each theory professor to a security professor (4 choices for the first, 3 for the second and so on).

## Task 7 – Lizards and Snakes!

Loudon has three pet lizards (Rango, a gecko named Gordon, and a goanna named Joanna) as well as two small pet snakes (Kaa and Basilisk) but only 4 terrariums to put them in. In how many different ways can he put his 5 pets in these 4 terrariums so that no terrarium has both a snake and a lizard?

> Find the number of possibilities when Kaa and Basilisk go in the same terrarium, and find the number of possibilities when they go to different terrariums, and then use the sum rule to get the final answer.
>
> If Kaa and Basilisk go in the same terrarium, there are 4 terrarium it could be. For each such choice, there are 3 choices of terrarium for each of the 3 lizards, so $3^3$ choices for all the lizards.
>
> If Kaa and Basilisk go in different terrariums, there are $4 \cdot 3 = 12$ pairs of terrariums they could go in. For each such choice, there are 2 choices of terrarium for each of the 3 lizards, so $2^3$ choices for all the lizards.
>
> Therefore the answer is $\boxed{4 \cdot 3^3 + 12 \cdot 2^3}$.

## Task 8 – Birthday Cake

A chef is preparing desserts for the week, starting on a Sunday. On each day, only one of five desserts (apple pie, cherry pie, strawberry pie, pineapple pie, and cake) may be served. On Thursday there is a birthday, so cake must be served that day. On no two consecutive days can the chef serve the same dessert. How many dessert menus are there for the week?

> Apply the product rule. Start from Thursday and work forward and backward in the week:
>
> More precisely, given the 1 choice on Thursday, for each of Wednesday and Friday, there are 4 choices (the different pie options). Given the choice on Wednesday, there are 4 choices for Tuesday, and given the choice on Tuesday, there are 4 choices for Monday, and given the choice on Monday, there are 4 choices on Sunday. Similarly, given the choice on Friday, there are 4 choices on Saturday.
>
> Therefore the answer is $4 \cdot 4 \cdot 4 \cdot 4 \cdot 1 \cdot 4 \cdot 4 = \boxed{4^6}$

## Task 9 – Photographs

Suppose that 8 people, including you and a friend, line up for a picture. In how many ways can the photographer organize the line if she wants to have fewer than 2 people between you and your friend?

> This is most easily solved using the sum rule. Count the number of ways the line can be organized if you are next to your friend. Then count the number of ways the line can be organized if there is one person between you and your friend. Then use the sum rule to add these up.
>
> **Case 1**: You are next to your friend. So we can think of you and your friend as being a "unit". Now apply the product rule: there are 7! ways to arrange the other 6 people together with the unit (of you and your friend). Once arranged, there are 2 ways to rearrange you and your friend in the order. So there are $7! \cdot 2$ ways to to line people up if you are next to your friend.
>
> **Case 2**: There is exactly 1 person between you and your friend. Apply the product rule by first picking the person who is between you (6 choices). Then, thinking of you, your friend and that person as a "unit", consider all arrangements of the 5 people plus the unit (6! ways). Finally, there are two ways for you and your friend to be placed within the trio. Therefore, altogether there are $6 \cdot 6! \cdot 2$ possibilities.
>
> Therefore, the final answer is $\boxed{(2 \cdot 7 + 2 \cdot 6) \cdot 6!}$

## Task 10 – Extended Family Portrait

A group of $n$ families, each with $m$ members, are to be lined up for a photograph. In how many ways can the $nm$ people be arranged if members of a family must stay together?

> Apply the product rule. First order the families; there are $n!$ ways to do this. Then consider the families one by one and reorder their members. Within each family, there are $m!$ ways to order their members. So there are a total of $\boxed{n!(m!)^n}$ ways to line these people up according to the given constraints.

*The material for the following questions has not yet been covered in lecture, but you may find them useful references for the homework.*

## Task 11 – HBCDEFGA

How many ways are there to permute the 8 letters A, B, C, D, E, F, G, H so that A is not at the beginning and H is not at the end?

> We can broadly approach this problem using complementary counting, where the number of permutations where A is not at the beginning and H is not at the end is the same as counting the total number of permutations − the permutations with A at the beginning or H at the end.
>
> The total number of permutations of 8 letters is $8!$.

Let's define the following notation: $B$ is the set of permutations with A at the beginning. $E$ is the set of permutations with H at the end.

We are looking for $|B \cup E|$, which is equal to $|B| + |E| - |B \cap E|$ by inclusion-exclusion. $|B|$, the number of permutations with $A$ at the beginning is $7!$ and $|E|$, the number with $H$ at the end is $7!$. Since there are $6!$ permutations that have $A$ at the beginning and $H$ at the end, $|B \cap E| = 6!$. By inclusion-exclusion, the number that have either $A$ at the beginning or $H$ at the end or both is $2 \cdot 7! - 6!$.

Finally, using complementary counting, the number that have neither $A$ at the end or $H$ at the end is $8! - (2 \cdot 7! - 6!)$.

## Task 12 – Binomial Theorem

What is the coefficient of $z^{36}$ in $(-2x^2yz^3 + 5uv)^{312}$?

By the Binomial Theorem,

$$(-2x^2yz^3+5uv)^{312} = \sum_{k=0}^{312} \binom{312}{k} (-2x^2yz^3)^k (5uv)^{312-k} = \sum_{k=0}^{312} \binom{312}{k} (-2)^k x^{2k} y^k z^{3k} (5uv)^{312-k}$$

The term that gives $z^{36}$ is the one with $k = 12$. Therefore, the coefficient is $\boxed{\binom{312}{12}(-2x^2y)^{12}(5uv)^{300}.}$

## Task 13 – Multinomial Coefficients

How many ways can we arrange the letters in 'TEDDYBEAR'?

There are $9$ letters in the word, of which $7$ are distinct letters (the set: $\{T, E, D, Y, B, A, R\}$). The groups of distinct letters have the following sizes: there are $2$ E's, $2$ D's and $1$ of each of the remaining letters. Using multinomial coefficients, we know the number of arrangements of letters is:

$$\binom{9}{2,2,1,1,1,1,1} = \frac{9!}{2! \cdot 2! \cdot 1! \cdot 1! \cdot 1! \cdot 1! \cdot 1!} = \frac{9!}{2! \cdot 2!}$$