Section 1

Review

- **Sum rule.** If you can choose from EITHER one of n options, OR one of m options with NO overlap with the previous n, then the number of possible outcomes of the experiment is $n + m$

- **Product rule.** In a sequential process with m steps, if there are n_1 choices for the 1st step, n_2 choices for the 2nd step (given the first choice), ..., and n_m choices for the mth step (given the previous choices), then the total number of outcomes is $n_1 n_2 ... n_m$

- **Number of ways to order n distinct objects:** $n! = n \cdot (n-1) \cdots 3 \cdot 2 \cdot 1$

- **Number of ways to select from n distinct objects:**
  - **Permutations** (number of ways to linearly arrange k objects out of n distinct objects, when the order of the k objects matters):
  $$P(n,k) = \frac{n!}{(n-k)!}$$
  - **Combinations** (number of ways to choose k objects out of n distinct objects, when the order of the k objects does not matter):
  $$\frac{n!}{k!(n-k)!} = \binom{n}{k} = C(n,k)$$

- **Complementary Counting:** If asked to find the number of ways to do X, you can: (1) find the total number of ways to do everything and then (2) subtract the number of ways to *not* do X.

  The rest of these will be covered in class on Friday.

- **Binomial theorem.** $\forall x, y \in \mathbb{R}, \forall n \in \mathbb{N}$: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$

- **Principle of Inclusion-Exclusion (PIE):** 2 events: $|A \cup B| = |A| + |B| - |A \cap B|$
  3 events: $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$
  In general: +singles - doubles + triples - quads + ...

- **Counting when there are repeats - multinomial coefficients.** Suppose there are n objects, but only k are distinct, with $k \leqslant n$. (For example, "godoggy" has $n = 7$ objects (characters) but only $k = 4$ are distinct: (g, o, d, y)). Let n_i be the number of times object i appears, for $i \in \{1, 2, \ldots, k\}$. (For example, $(3, 2, 1, 1)$, continuing the "godoggy" example.) The number of distinct ways to arrange the n objects is $\binom{n}{n_1, \ldots, n_k} = \frac{n!}{n_1! \cdot \ldots \cdot n_k!}$

Task 1 – Sets

(a) For each one of the following sets, give its **cardinality**, i.e., indicate how many elements it contains:

- $A = \varnothing$
- $B = \{\varnothing\}$
- $C = \{\{\varnothing\}\}$
- $D = \{\varnothing, \{\varnothing\}\}$

(b) Let $S = \{a, b, c\}$ and $T = \{c, d\}$. Compute:

- $S \cup T$
- $S \cap T$
- $S \setminus T$
- $2^{S \setminus T}$
- $S \times T$

Task 2 – Basic Counting

a) Credit-card numbers are made of 15 decimal digits, and a 16th checksum digit (which is uniquely determined by the first 15 digits). How many credit-card numbers are there?

b) How many positive divisors does $1440 = 2^5 3^2 5$ have?

c) How many ways are there to arrange the CSE 312 staff on a line (11 TAs, two professors) for a group picture?

d) How many ways are there to arrange the CSE 312 staff on a line so that Professors Tessaro and Beame are at the two ends of the line?

Task 3 – Seating

How many ways are there to seat 10 people, consisting of 5 couples, in a row of 10 seats if ...

a) ... all couples are to get adjacent seats?

b) ... anyone can sit anywhere, except that one couple insists on *not* sitting in adjacent seats?

Task 4 – Weird Card Game

In how many ways can a pack of fifty-two cards (in four suits of thirteen cards each) be dealt to thirteen players, four to each, so that every player has one card from each of the suits?

Task 5 – Full Class

There are 40 seats and 40 students in a classroom. Suppose that the front row contains 10 seats, and there are 5 students who must sit in the front row in order to see the board clearly. How many seating arrangements are possible with this restriction?

Task 6 – Escape the Professor

There are 6 security professors and 7 theory professors taking part in an escape room. The solution requires that they choose 4 pairs, each consisting of one security professor and one theory professor. How many options for pairings do they have?

Task 7 – Lizards and Snakes!

Loudon has three pet lizards (Rango, a gecko named Gordon, and a goanna named Joanna) as well as two small pet snakes (Kaa and Basilisk) but only 4 terrariums to put them in. In how many different ways can he put his 5 pets in these 4 terrariums so that no terrarium has both a snake and a lizard?

Task 8 – Birthday Cake

A chef is preparing desserts for the week, starting on a Sunday. On each day, only one of five desserts (apple pie, cherry pie, strawberry pie, pineapple pie, and cake) may be served. On Thursday there is a birthday, so cake must be served that day. On no two consecutive days can the chef serve the same dessert. How many dessert menus are there for the week?

Task 9 – Photographs

Suppose that 8 people, including you and a friend, line up for a picture. In how many ways can the photographer organize the line if she wants to have fewer than 2 people between you and your friend?

Task 10 – Extended Family Portrait

A group of n families, each with m members, are to be lined up for a photograph. In how many ways can the nm people be arranged if members of a family must stay together?

The material for the following questions has not yet been covered in lecture, but you may find them useful references for the homework.

Task 11 – HBCDEFGA

How many ways are there to permute the 8 letters A, B, C, D, E, F, G, H so that A is not at the beginning and H is not at the end?

Task 12 – Binomial Theorem

What is the coefficient of z^{36} in $(-2x^2yz^3 + 5uv)^{312}$?

Task 13 – Multinomial Coefficients

How many ways can we arrange the letters in 'TEDDYBEAR'?