Section 6

CLT and Joint Distributions

CONTENT REVIEW

normal distribution

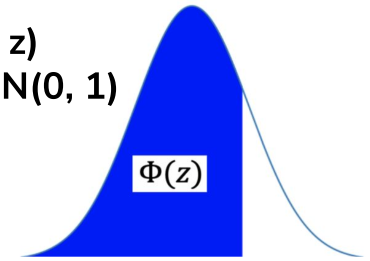
this PDF is really messy :(

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

use the Phi Table to easily look up the value of P(Z < z)

1. **Find** μ)/σ^2 of normal RV X
1. **Standardize** RV X~N(μ, σ^2) to get (X-μ)/σ^2 ~ Z ~ N(0,1)
2. Use **Phi table** to get appropriate value with Z =(X-μ)/σ^2
3. **Solve** for X

P(Z < z)
if Z ~ N(0, 1)



normal distribution

properties of the normal distribution

Closure for Normal Distribution

Let $X \sim N(\mu, \sigma^2)$. Then $aX + b \sim N(a\mu + b, b^2\sigma^2)$

"Reproductive" Property

Let X1,..,Xn be <u>independent</u> normal RVs with E[Xi]=μ and Var(Xi=σi

$$X = \sum_{i=1}^{n}(a_iX_i + b) \sim N\left(\sum_{i=1}^{n}(a_i\mu_i + b), \sum_{i=1}^{n} a_i^2\sigma_i^2\right)$$

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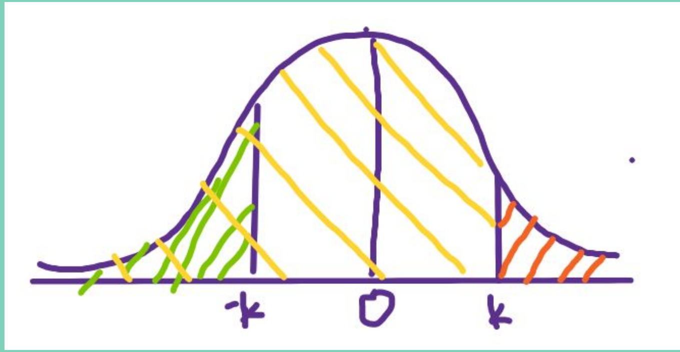

- By symmetry of normal distribution, $P(X < -k) = P(X > k)$
- $\Phi(a) = P(Z < a)$ where Z (standard normal) follows $N(0, 1)$
- $\Phi(-k) = P(Z < k) = P(Z>k) = 1-P(Z <k) = 1-\Phi(k)$

  ^ *this is useful because negative values aren't on the table!*

it turns out that a lot of data collected in real world experiments follow a normal distribution!

central limit theorem

if $X = X_1 + X_2 + \ldots + X_n$ are **iid** *where* **E[Xi] = μ** *and* ***Var(Xi) = σ²***,
X can be approximated by a normal distribution **$X \sim N(n\mu, n\sigma^2)$**
as n becomes really large

this also means that as n increases, $(X-\mu / \sigma^{2)}) \sim Z \sim N(0,1)$

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if Xi is discrete random variables, use **continuity correction**

because we'd be using a continuous distribution to approximate a discrete distribution

To estimate probability that discrete RV lands in (integer) interval {a,b}
compute probability continuous approximation lands in interval [a-0.5, b+0.5]

central limit theorem

1. set up the problem
(what do we want to solve for/what is the probability we want to be true?)

2. if the distribution is discrete, apply continuity correction here

2. apply CLT to $X = X_i + \ldots + X_n$

3. convert to standard normal

4. solve!

joint distribution

shows the distribution for two (or more) random variables

$$p_{X,Y}(x, y) = P(X=x, Y=y)$$

	Discrete	Continuous
Joint PMF/PDF	$p_{X,Y}(x,y) = \mathbb{P}(X=x, Y=y)$	$f_{X,Y}(x,y) \neq \mathbb{P}(X=x, Y=y)$
Joint range/support $\Omega_{X,Y}$	$\{(x,y) \in \Omega_X \times \Omega_Y : p_{X,Y}(x,y) > 0\}$	$\{(x,y) \in \Omega_X \times \Omega_Y : f_{X,Y}(x,y) > 0\}$
Joint CDF	$F_{X,Y}(x,y) = \sum_{t \leqslant x, s \leqslant y} p_{X,Y}(t,s)$	$F_{X,Y}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(t,s)\, ds dt$
Normalization	$\sum_{x,y} p_{X,Y}(x,y) = 1$	$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx dy = 1$
Marginal PMF/PDF	$p_X(x) = \sum_y p_{X,Y}(x,y)$	$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) dy$
Expectation	$\mathbb{E}[g(X,Y)] = \sum_{x,y} g(x,y) p_{X,Y}(x,y)$	$\mathbb{E}[g(X,Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y) dx dy$
Independence must have	$\forall x, y, p_{X,Y}(x,y) = p_X(x) p_Y(y)$ $\Omega_{X,Y} = \Omega_X \times \Omega_Y$	$\forall x, y, f_{X,Y}(x,y) = f_X(x) f_Y(y)$ $\Omega_{X,Y} = \Omega_X \times \Omega_Y$
Conditional PMF/PDF	$p_{X\|Y}(x\|y) = \frac{p_{X,Y}(x,y)}{p_Y(y)}$	$f_{X\|Y}(x\|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$
Conditional Expectation	$\mathbb{E}[X\|Y=y] = \sum_x x \cdot p_{X\|Y}(x\|y)$	$\mathbb{E}[X\|Y=y] = \int_{-\infty}^{\infty} x f_{X\|Y}(x\|y) dx$

Suppose X and Y have the following joint PMF:

X/Y	1	2	3
0	0	0.2	0.1
1	0.3	0	0.4

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the values in the table show the joint probabilities - for example this highlighted value is P(X=1, Y-3)

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(a) Identify the range of X (Ω_X), the range of Y (Ω_Y), and their joint range ($\Omega_{X,Y}$).

- Based on the table (specifically what's highlighted in yellow and pink), we know that the range of X is {0, 1}, and the range of Y is {1,2,3}

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- The joint range is the set of *pairs of x and y* that both X and Y can take on at the same time. In other word the pairs of values x and y such that the joint PMF P(X=x, Y=y) is greater than 0.

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- In this case, that would be the set of pairs:

  $\Omega_{X,Y}$ = {(0, 2), (0,3), (1,1), (1,3)}
- We ***don't*** include the pairs (0, 1) and (1,2) because the joint pmf is 0 for those pairs

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1	0.3	0	0.4

(b) Find the marginal PMF for X, $p_X(x)$ for $x \in \Omega_X$.

- The marginal PMF for X is just like asking what the PMF for X is based on the joint PMF
- There are only two values in the range of X so we can try defining this PMF by looking at the probabilities for each value separately!
- First, we want to find P(X=0) -

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  P(X=0) = P(X=0, Y=1)+P(X=0,Y=2) + P(X=0,Y=3)=0+0.2+0.1-0,3

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- Similarly, P(X=1) = 0.3+0+0.4
- <u>In general, the marginal pmf looks like:</u> $p_X(x) = \sum_{y \in \Omega_Y} p_{X,Y}(x,y))$

Suppose X and Y have the following joint PMF:

X/Y	1	2	3
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(d) Are X and Y independent? Why or why not?

- If we want to prove that X and Y are independent, we would need to show both of the conditions to be true:

$$\forall x, y, p_{X,Y}(x,y) = p_X(x) p_Y(y)$$
$$\Omega_{X,Y} = \Omega_X \times \Omega_Y$$

- However, note that the second condition is not true in this case! (take a look at the ranges we found in part a). So, X and Y are not independent.

Thanks for coming today!