312 Midterm Review Session

Summer 2024

Slides made by Cleah Winston

Midterm Info

When & Where?

Monday, July 15th, 3:30-5:20pm, in BAG 131

What to Bring?

Writing tool (pencil, pen, eraser)

Photo id (e.g., Husky card)

A4-sized paper of notes

How to Prepare?

Practice exams on course website

Section handouts

Review old homework

Lecture slides & notes

Midterm Topics

- Counting
- Pigeonhole Principle
- Binomial Theorem
- Combinatorial Proofs
- Probability Basics
- Uniform Probability Spaces
- Conditional Probability
- Independence
- Random Variables
- Linearity of Expectation

Midterm Reference Sheet

Reference Sheet

Permutations and Combinations
Select k distinct elements from n distinct items.
If we care about the order, use a permutation: $P(n, k)$
If we don't care about the order, use a combination: $\binom{n}{k}$

Binomial Theorem
Let $x, y \in \mathbb{R}$ and $n \in \mathbb{N}$ positive integer. Then: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$.

Principle of Inclusion-Exclusion (PIE)
2 events: $|A \cup B| = |A| + |B| - |A \cap B|$
3 events: $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$
k events: singles + doubles + triples - quads + ...

Stars and Bars
If we have n indistinguishable objects being split up into k distinguishable groups/types, there are $\binom{n+k-1}{k-1} = \binom{n+k-1}{n}$ options.

Pigeonhole Principle
If there are n pigeons we want to put into k holes (where $n > k$), then at least one pigeonhole must contain at least 2 (or to be precise, $\lceil n/k \rceil$) pigeons.

Key Probability Definitions
The sample space is the set Ω of all possible outcomes of an experiment. An event is any subset $E \subseteq \Omega$. Events E and F are mutually exclusive if $E \cap F = \emptyset$.

Law of Total Probability (LTP)
If events $E_1, \ldots, E_n$ partition Ω, then for any event F:
$P(F) = \sum_{i=1}^{n} P(F \cap E_i) = \sum_{i=1}^{n} P(F|E_i)P(E_i)$

Bayes Theorem with LTP
Let events $E_1, E_2, \ldots, E_n$ partition the sample space Ω, and let F be another event. Then:
$P(E_i|F) = \frac{P(F|E_i)P(E_i)}{\sum_{i=1}^{n} P(F|E_i)P(E_i)}$

Independence (Events)
A and B are independent if any of the following equivalent statements hold:

1. $P(A \cap B) = P(A)P(B)$
2. $P(A|B) = P(A)$
3. $P(B|A) = P(B)$

Chain Rule
Let $A_1, \ldots, A_n$ be events with nonzero probabilities. Then:
$P(A_1 \cap A_2 \cap \ldots \cap A_n) = P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2) \cdots P(A_n|A_1 \cap \ldots \cap A_{n-1})$

Mutual Independence (Events)
We say n events $A_1, A_2, \ldots, A_n$ are (mutually) independent if, for any subset $I \subseteq [n] = \{1, 2, \ldots, n\}$, we have $P\left(\bigcap_{i \in I} A_i\right) = \prod_{i \in I} P(A_i)$
This equation is actually representing 2^n equations since there are 2^n subsets of $[n]$.

Conditional Independence
A and B are conditionally independent given an event

an experiment. An event is any subset $E \subseteq \Omega$. Events E and F are mutually exclusive if $E \cap F = \emptyset$.

Probability Space
A probability space is a pair $(\Omega, \mathcal{P})$, where Ω is the sample space and $\mathcal{P}$ is a probability measure assigning probabilities to each outcome in the sample space, such that $\sum_{\omega \in \Omega} \mathcal{P}(\omega) = 1$. The probability of an event $E \subseteq \Omega$ is $\mathcal{P}(E) = \sum_{\omega \in E} P(e)$.

Definition of Conditional Probability
$P(A|B) = \frac{P(A \cap B)}{P(B)}$

Bayes' Theorem
$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$

Definition: Partition
Non-empty events $E_1, \ldots, E_n$ partition the sample space Ω if:

- **Exhaustive:** $E_1 \cup E_2 \cup \cdots \cup E_n = \Omega$ (they cover the entire sample space).
- **Pairwise Mutually Exclusive:** For all $i \neq j$, $E_i \cap E_j = \emptyset$ (none of them overlap)

Conditional Independence
A and B are conditionally independent given an event C if any of the following equivalent statements hold:

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Random Variable (RV)
A random variable (RV) X is a numeric function of the outcome $X : \Omega \to \mathbb{R}$. The set of possible values X can take on is its range/support, denoted Ω_X.

Probability Mass Function (PMF)
For a discrete RV X, it assigns probabilities to values in its range. That is $p_X : \Omega_X \to [0, 1]$, where $p_X(k) = P(X = k)$.

Expectation
The expectation of a discrete RV X is: $\mathbb{E}[X] = \sum_{k \in \Omega_X} k \cdot p_X(k)$.

Linearity of Expectation (LoE)
For any random variables X, Y (possibly dependent):
$\mathbb{E}[aX + bY + c] = a\mathbb{E}[X] + b\mathbb{E}[Y] + c$

Midterm Reference Sheet

Picking from a group vs. ordering

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IF YOU SEE "**OR**"

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3 events: $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$
k events: singles + doubles + triples ...

IF YOU SEE "**IDENTICAL**"

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If we have n indistinguishable objects being split up into k distinguishable groups/types, there are $\binom{n+k-1}{k-1} = \binom{n+k-1}{n}$ options.

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IF YOU SEE "**if...then**"

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For any random variables X, Y (possibly dependent): $\mathbb{E}[aX + bY + c] = a\mathbb{E}[X] + b\mathbb{E}[Y] + c$

Practice Midterm #4

Find it on course website or grab a physical copy from the front!

Q1: A Variance of Topics

Kahoot!

Q2a: I Love Bubble Tea

a) Marlene wants to buy 20 bubble teas for a party. There are 4 flavors available- brown sugar, matcha, lychee, and strawberry. If she wants to buy at least 2 of each flavor, and if bubble teas of the same flavor are indistinguishable, how many ways are there to select the 20 drinks?

Q2a: I Love Bubble Tea

a) Marlene wants to buy 20 bubble teas for a party. There are 4 flavors available- brown sugar, matcha, lychee, and strawberry. If she wants to buy at least 2 of each flavor, and if bubble teas of the same flavor are indistinguishable, how many ways are there to select the 20 drinks?

Stars and Bars
If we have n indistinguishable objects being split up into k distinguishable groups/types, there are $\binom{n+k-1}{k-1} = \binom{n+k-1}{n}$ options.

Q2b: I Love Bubble Tea

b) Now, Marlene goes to a different bubble tea shop where each drink is a combination of 4 distinct flavors. In this shop, there are 10 different flavors (two of which are brown sugar and matcha). How many ways can Marlene order 1 drink, such that the drink she ordered contains brown sugar flavor, and/or contains matcha flavor? Consider two orders the same they have the same combination of flavors.

b) Now, Marlene goes to a different bubble tea shop where each drink is a combination of 4 distinct flavors. In this shop, there are 10 different flavors (two of which are brown sugar and matcha). How many ways can Marlene order 1 drink, such that the drink she ordered contains brown sugar flavor, and/or contains matcha flavor? Consider two orders the same they have the same combination of flavors.

Q2b: I Love Bubble Tea

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3 events: $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

k events: singles + doubles + triples - quads + ...

Q3a: Society of Cards

Suppose we have a deck of cards consisting of the 4 suits, each with cards ranked from 1 to 13, plus 2 distinguishable Joker cards (Joker A, Joker B). The deck has (13 * 4)+2 = 54 cards total. Suppose you draw a hand of 5 cards from this deck, with each hand equally likely to be drawn.

a) What is the probability of your hand containing at least 1 Joker?

Q3b: Society of Cards

Suppose we have a deck of cards consisting of the 4 suits, each with cards ranked from 1 to 13, plus 2 distinguishable Joker cards (Joker A, Joker B). The deck has (13 * 4)+2 = 54 cards total. Suppose you draw a hand of 5 cards from this deck, with each hand equally likely to be drawn.

b) Let J be the event that your hand contains exactly 1 Joker. Let S be the event that your hand contains the "1 of Spades" ("1♠") card. Find P(J U S).

Q3c: Society of Cards

Suppose we have a deck of cards consisting of the 4 suits, each with cards ranked from 1 to 13, plus 2 distinguishable Joker cards (Joker A, Joker B). The deck has (13 * 4)+2 = 54 cards total. Suppose you draw a hand of 5 cards from this deck, with each hand equally likely to be drawn.

c) What is the probability that your hand is a "Straight", that is, a hand where the cards' ranks form a consecutive sequence?

Q4: Superstition or Reality?

Allie and her daughter Sophia love watching Kraken ice hockey games. For some games, one of them will wear the lucky tentacle, which affects the Kraken's chances of winning.

- If neither wears the lucky tentacle, the Kraken win with probability 1/2.
- If Allie wears the lucky tentacle, the Kraken win with probability p (you'll solve for the constant p later).
- If Sophia wears the lucky tentacle, the Kraken win with probability 2p.

Note that these are the only possibilities (there is no way for both to wear the one lucky tentacle). Let A, S, N be the events that Allie, Sophia, or neither wear the lucky tentacle (respectively). Let W be the event that the Kraken win.

Q4a: Superstition or Reality?

Allie and her daughter Sophia love watching Kraken ice hockey games. For some games, one of them will wear the lucky tentacle1, which affects the Kraken's chances of winning.

- **If neither wears the lucky tentacle, the Kraken win with probability 1/2.**
- If Allie wears the lucky tentacle, the Kraken win with probability p (you'll solve for the constant p later).
- If Sophia wears the lucky tentacle, the Kraken win with probability 2p.

Note that these are the only possibilities (there is no way for both to wear the one lucky tentacle). Let A, S, N be the events that Allie, Sophia, or neither wear the lucky tentacle (respectively). Let W be the event that the Kraken win.

a) Which conditional probability or probability is referred to in the first bullet?

Q4b: Superstition or Reality?

Allie and her daughter Sophia love watching Kraken ice hockey games. For some games, one of them will wear the lucky tentacle1, which affects the Kraken's chances of winning.

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b) Sophia has an early bed-time, so she only wears the tentacle 1/5 of the time. Allie wears it 2/5 of the time and the remaining 2/5 neither do. We also know that the Kraken won 13 25 of their games. Write an expression that will let you solve for p, in terms of only p and numerical values.

Q4c: Superstition or Reality?

Allie and her daughter Sophia love watching Kraken ice hockey games. For some games, one of them will wear the lucky tentacle1, which affects the Kraken's chances of winning.

- If neither wears the lucky tentacle, the Kraken win with probability 1/2.
- If Allie wears the lucky tentacle, the Kraken win with probability p (you'll solve for the constant p later).
- If Sophia wears the lucky tentacle, the Kraken win with probability 2p.

Note that these are the only possibilities (there is no way for both to wear the one lucky tentacle). Let A, S, N be the events that Allie, Sophia, or neither wear the lucky tentacle (respectively). Let W be the event that the Kraken win.

c) Now, solve that formula for p.

Q4d, e: Superstition or Reality?

Allie and her daughter Sophia love watching Kraken ice hockey games. For some games, one of them will wear the lucky tentacle1, which affects the Kraken's chances of winning.

- If neither wears the lucky tentacle, the Kraken win with probability 1/2.
- If Allie wears the lucky tentacle, the Kraken win with probability p (you'll solve for the constant p later).
- If Sophia wears the lucky tentacle, the Kraken win with probability 2p.

Note that these are the only possibilities (there is no way for both to wear the one lucky tentacle). Let A, S, N be the events that Allie, Sophia, or neither wear the lucky tentacle (respectively). Let W be the event that the Kraken win.

d) If the Kraken won, what is the probability that Sophia was wearing the lucky tentacle? Write a formula to represent this probability using only notation (no numbers nor expressions using p; just probabilities, conditional probabilities, and events) that describe the situation.

e) Now plug in numbers/expressions in terms of p for each of the parts in the formula you wrote in part (d). You do not need to simplify, but may if you think it would help you check your work.

Q5a: Robbie Can't Sing

A group of 5 friends visits a karaoke lounge. The karaoke machine has 20 songs. The friends share the same opinion on every song, so of the available songs: 12 are songs they like, while the remaining 8 are ones they dislike. The group sets the playlist on shuffle, such that every ordering of the 20 songs is equally likely. They decide to take turns singing 4 (consecutive) songs each. A person is unhappy if they dislike all 4 songs they sing.

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a) What is the probability that the first person to sing is unhappy?

Q5b: Robbie Can't Sing

A group of 5 friends visits a karaoke lounge. The karaoke machine has 20 songs. The friends share the same opinion on every song, so of the available songs: 12 are songs they like, while the remaining 8 are ones they dislike. The group sets the playlist on shuffle, such that every ordering of the 20 songs is equally likely. They decide to take turns singing 4 (consecutive) songs each. A person is unhappy if they dislike all 4 songs they sing.

b) What is the expected number of unhappy friends once everyone is done singing?

Q5c: Robbie Can't Sing

A group of 5 friends visits a karaoke lounge. The karaoke machine has 20 songs. The friends share the same opinion on every song, so of the available songs: 12 are songs they like, while the remaining 8 are ones they dislike. The group sets the playlist on shuffle, such that every ordering of the 20 songs is equally likely. They decide to take turns singing 4 (consecutive) songs each. A person is unhappy if they dislike all 4 songs they sing.

Now, suppose the 20th song is followed by the 1st song. The shuffle order is the same when looping back to the first song that was played. This means that every song i (where 1 i 20) is always immediately preceded by one song and immediately followed by another song. Let Xk be the number of liked songs played immediately before or immediately after the kth disliked song (recall that each song has exactly one "immediately before" and "immediately following" it).

c) What is Ω_Xk , i.e. the support of Xk?

Q5d: Robbie Can't Sing

A group of 5 friends visits a karaoke lounge. The karaoke machine has 20 songs. The friends share the same opinion on every song, so of the available songs: 12 are songs they like, while the remaining 8 are ones they dislike. The group sets the playlist on shuffle, such that every ordering of the 20 songs is equally likely. They decide to take turns singing 4 (consecutive) songs each. A person is unhappy if they dislike all 4 songs they sing.

Now, suppose the 20th song is followed by the 1st song. The shuffle order is the same when looping back to the first song that was played. This means that every song i (where 1 i 20) is always immediately preceded by one song and immediately followed by another song. Let Xk be the number of liked songs played immediately before or immediately after the kth disliked song (recall that each song has exactly one “immediately before” and “immediately following” it).

d) Find the PMF of Xk.

Q5e: Robbie Can't Sing

A group of 5 friends visits a karaoke lounge. The karaoke machine has 20 songs. The friends share the same opinion on every song, so of the available songs: 12 are songs they like, while the remaining 8 are ones they dislike. The group sets the playlist on shuffle, such that every ordering of the 20 songs is equally likely. They decide to take turns singing 4 (consecutive) songs each. A person is unhappy if they dislike all 4 songs they sing.

Now, suppose the 20th song is followed by the 1st song. The shuffle order is the same when looping back to the first song that was played. This means that every song i (where 1 i 20) is always immediately preceded by one song and immediately followed by another song. Let Xk be the number of liked songs played immediately before or immediately after the kth disliked song (recall that each song has exactly one "immediately before" and "immediately following" it).

e) Find the expected number of liked songs played immediately before or immediately after a disliked song using your answer to the previous question.

Q5f: Robbie Can't Sing

A group of 5 friends visits a karaoke lounge. The karaoke machine has 20 songs. The friends share the same opinion on every song, so of the available songs: 12 are songs they like, while the remaining 8 are ones they dislike. The group sets the playlist on shuffle, such that every ordering of the 20 songs is equally likely. They decide to take turns singing 4 (consecutive) songs each. A person is unhappy if they dislike all 4 songs they sing.

A switch occurs when a liked song is immediately next to a disliked song. For example, consider the following sequences:

- Disliked Disliked Liked- this sequence contains 1 switch
- Liked Disliked Liked- this sequence contains 2 switches

f) What is the expected number of times we switch between disliked and liked songs if we play 21 songs (the 20 available songs, followed by the one we started with)? You may use the variable e to represent the answer to part (e) in your answer.

Questions?

Thank you for coming!

Recording and slides will be posted.

Slides made by Cleah Winston