Linearity of Expectation

CSE 312 24Su

Lecture 9

Outline

Last time, we introduced random variables (RVs)
function that assign a quantitative value to an outcome of a random experiment

- Describe RVs with things like the support, PMF, CDF
- Expected value of a RV is like the “average” value it takes on

Today…

> **Independence** of random variables

> **Expectation of a function** of a random variable (e.g., $E[X^2]$)

> **Linearity of expectation**

Statement
Proof
A whole bunch of examples

Independence of Random Variables

Independence of events

Recall the definition of independence of **events**:

Independence

Two events A, B are independent if

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$$

"knowing whether one event occurred doesn't tell us anything about whether the other event occurred"

Independence of Random Variables

That's for events…what about random variables?

Independence (of random variables)

X and Y are independent if for all k, ℓ

$$\mathbb{P}(X = k, Y = \ell) = \mathbb{P}(X = k)\mathbb{P}(Y = \ell)$$

We'll often use commas instead of ∩ symbol to save space.

"knowing the value of one random variable doesn't tell us anything about what the value of the other might be"

Independence of Random Variables

The “for all values” is important.

We say that the event “the sum is 7” is independent of “the red die is 5”

What about $S =$“the sum of two dice” and $R =$“the value of the red die”

Independence of Random Variables

The "for all values" is important.

We say that the event "the sum is 7" is independent of "the red die is 5"

What about $S =$"the sum of two dice" and $R =$"the value of the red die"

NOT independent.

$\mathbb{P}(S = 2, R = 5) \neq \mathbb{P}(S = 2)\mathbb{P}(R = 5)$ (for example)

Independence of Random Variables

Flip a coin independently $2n$ times.

Let X be “the number of heads in the first n flips.”

Let Y be “the number of heads in the last n flips.”

X and Y are independent.

Mutual Independence for RVs

A little simpler to write down than for events

Mutual Independence (of random variables)

$X_1, X_2, \ldots, X_n$ are mutually independent if for all $x_1, x_2, \ldots, x_n$

$$\mathbb{P}(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n) = \mathbb{P}(X_1 = x_1)\mathbb{P}(X_2 = x_2) \cdots \mathbb{P}(X_n = x_n)$$

DON'T need to check all subsets for random variables…

But you do need to check all values (all possible x_i) still.

Expectation of a *Function* of a Random Variable

Expectation

Expectation

The "expectation" (or "expected value") of a random variable X is:

$$\mathbb{E}[\boldsymbol{X}] = \sum_{\boldsymbol{k} \in \Omega_X} \boldsymbol{k} \cdot \mathbb{P}(\boldsymbol{X} = \boldsymbol{k})$$

$$\mathbb{E}[\boldsymbol{X}] = \sum_{\omega \in \Omega} X(\omega) \cdot \mathbb{P}(\omega)$$

Intuition: The weighted average of values X could take on.

Weighted by the probability you actually see them.

What about $\mathbb{E}[g(X)]$? (e.g., $\mathbb{E}[X^2]$, $\mathbb{E}[2^X]$)

Applying **functions** on a random variable(s).

$g(X) = 2X + 3$

$g(X) = X^2$

$g(X) = 2^X$

$g(X, Y) = X + Y$

Still gives us a random variable!

Given an outcome, these functions give you a number.

They're functions from $\Omega \rightarrow \mathbb{R}$. That's the definition of a random variable!

What about $\mathbb{E}[g(X)]$? (e.g., $\mathbb{E}[X^2]$, $\mathbb{E}[2^X]$)

What if we want to find the expected value of some function of X?

Let's say we want to find $\mathbb{E}[X^2]$. Is $\mathbb{E}[X^2] = (\mathbb{E}[X])^2$?

Not necessarily! For example,

If we have a random variable X that following the PMF:

$$\mathrm{p}_X(k) = \begin{cases} 0.5 & k = 1 \\ 0.5 & k = -1 \\ 0 & \text{otherwise} \end{cases}$$

$\mathbb{E}[X] = 0.5 \cdot 1 + 0.5 \cdot -1 = 0 \rightarrow (\mathbb{E}[X])^2 = 0$

$\mathbb{E}[X^2] = 1$

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$$\mathbb{E}[X] = 0.5 \cdot 1 + 0.5 \cdot -1 = 0 \qquad \mathbb{E}[X^2] = 0.5 \cdot 1^{\mathbf{2}} + 0.5 \cdot (-1)^{\mathbf{2}} = 1$$

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Expectation of $g(X)$

“Law of the unconscious statistician” (LOTUS)

The “expectation” (or “expected value”) of $g(X)$ is:

$$\mathbb{E}[\boldsymbol{g}(\boldsymbol{X})] = \sum_{\boldsymbol{k}\in\Omega_X} \boldsymbol{g}(\boldsymbol{k}) \cdot \mathbb{P}(\boldsymbol{X} = \boldsymbol{k})$$

Exact same as formula for $E[X]$, but we **apply the function on each of the values in the support of $\boldsymbol{X}$** (the corresponding probabilities are the same)

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What if $\boldsymbol{g}(\boldsymbol{X})$ is a linear function? E.g., $\boldsymbol{g}(\boldsymbol{X}, \boldsymbol{Y}) = \boldsymbol{X} + \boldsymbol{Y}$

Linearity of Expectation

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For any two random variables X and Y:

$$\mathbb{E}[\boldsymbol{X}+\boldsymbol{Y}]=\mathbb{E}[\boldsymbol{X}]+\mathbb{E}[\boldsymbol{Y}]$$

Note: X and Y do not have to be independent

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Extending this to n random variables, $X_1, X_2, \dots, X_n$

$$\mathbb{E}[X_1 + X_2 + \cdots + X_n] = \mathbb{E}[X_1] + \mathbb{E}[X_2] + \cdots + \mathbb{E}[X_n]$$

This can be proven by induction.

Linearity of Expectation - Proof

Linearity of Expectation

For any two random variables X and Y:

$$\mathbb{E}[\boldsymbol{X}+\boldsymbol{Y}] = \mathbb{E}[\boldsymbol{X}] + \mathbb{E}[\boldsymbol{Y}]$$

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Proof:

$$\mathbb{E}[X+Y] = \Sigma_{\omega\in\Omega}\mathbb{P}(\omega)\big(X(\omega)+Y(\omega)\big)$$

Definition of expectation:

$$\mathbb{E}[\boldsymbol{X}] = \sum_{\omega\in\Omega} X(\omega)\cdot\mathbb{P}(\omega)$$

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Proof:

$$\mathbb{E}[X + Y] = \Sigma_{\omega \in \Omega} \mathbb{P}(\omega)\big(X(\omega) + Y(\omega)\big)$$
$$= \Sigma_{\omega \in \Omega} (\mathbb{P}(\omega) X(\omega) + \mathbb{P}(\omega) Y(\omega))$$

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Proof:

$$\begin{aligned}\mathbb{E}[X+Y] &= \Sigma_{\omega\in\Omega}\mathbb{P}(\omega)\big(X(\omega)+Y(\omega)\big)\\ &= \Sigma_{\omega\in\Omega}(\mathbb{P}(\omega)X(\omega)+\mathbb{P}(\omega)Y(\omega))\\ &= \Sigma_{\omega\in\Omega}\mathbb{P}(\omega)X(\omega)+\Sigma_{\omega\in\Omega}\mathbb{P}(\omega)Y(\omega)\end{aligned}$$

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&= \mathbb{E}[X]+\mathbb{E}[Y]
\end{aligned}$$

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Linearity of Expectation

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Constants are also fine:

For real numbers a, b, c

$$\begin{aligned}\mathbb{E}[aX + bY + c] &= \mathbb{E}[aX] + \mathbb{E}[bY + c] \\ &= a\mathbb{E}[X] + b\mathbb{E}[Y] + c\end{aligned}$$

Fishy Business

Say you and your friend go fishing everyday.

- You catch X fish, with $\mathbb{E}[X] = 3$
- Your friend catches Y fish, with $\mathbb{E}[Y] = 7$

- How many fish do both of you bring on an average day?

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$$\mathbb{E}[Z] = \mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y] = 3 + 7 = 10$$

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- You can sell each for \$10 per fish, but you need \$15 (total) for expenses. What is your average profit?

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- You can sell each for \$10 per fish, but you need \$15 (total) for expenses. What is your average profit?

$$\mathbb{E}[10Z - 15] = 10\mathbb{E}[Z] - 15 = 100 - 15 = 85$$

Coin Tosses

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Let Y be the r.v. representing the total number of heads

$$p_Y(y) = \begin{cases} \frac{1}{4} & \text{if } y = 0 \\ \frac{1}{2} & \text{if } y = 1 \\ \frac{1}{4} & \text{if } y = 2 \\ 0 & otherwise \end{cases}$$

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$$\mathbb{E}[Y] = \Sigma_{k \in \Omega_Y} p_Y(k) \cdot k = \frac{1}{4} \cdot 0 + \frac{1}{2} \cdot 1 + \frac{1}{4} \cdot 2 = 1$$

Repeated Coin Tosses

Now what if the probability of flipping a head was p and that we wanted to find the total number of heads flipped when we flip the coin n times?

Let X be the r.v. representing the total number of heads.

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Make a prediction --- what should $\mathbb{E}[X]$ be?

a) $n+p$
b) p^n
c) np
d) n/p

Fill out the poll everywhere:
pollev.com/cse312

Repeated Coin Tosses

Now what if the probability of flipping a head was p and that we wanted to find the total number of heads flipped when we flip the coin n times?

Let X be the r.v. representing the total number of heads.

$$\mathbb{E}[X] = \sum_{k=0}^{n} k \cdot \mathbb{P}(X = k) = \sum_{k=0}^{n} k \cdot \binom{n}{k} p^k (1-p)^{n-k}$$

Ok, but what actually is it?
I don't have intuition for this formula.

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$$= \sum_{k=1}^{n} k \cdot \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \sum_{k=1}^{n} n \cdot \binom{n-1}{k-1} p^k (1-p)^{n-k}$$

$$k\binom{n}{k} = n\binom{n-1}{k-1}$$

$$= np \sum_{i=0}^{n-1} \binom{n-1}{i} p^i (1-p)^{n-1-i}$$

$$= np(p + (1-p))^{n-1} = np$$

Binomial Theorem!

We did it! And all it took was a clever application of the binomial theorem, setup by a very non-obvious application of an obscure combinatorial identity. Ezpz.

Repeated Coin Tosses

Now what if the probability of flipping a head was p and that we wanted to find the total number of heads flipped when we flip the coin n times?

$$\mathbb{E}[X] = \sum_{k=0}^{n} k \cdot \mathbb{P}(Y = k) = \sum^{n} \ldots p)^{n-k}$$

$$= \sum^{n} \ldots$$

$$= \sum_{k=1} n \cdot \binom{n-1}{k-1} p^k (1-p)^{n-k}$$

$$= np \sum_{i=0}^{n-1} \binom{n-1}{i} p^i (1-p)^{n-1-i}$$

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No one wants to do proofs like this every time!

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For any two random variables X and Y:

$$\mathbb{E}[\boldsymbol{X}+\boldsymbol{Y}]=\mathbb{E}[\boldsymbol{X}]+\mathbb{E}[\boldsymbol{Y}]$$

Note: X and Y do not have to be independent

Extending this to n random variables, $X_1, X_2, \dots, X_n$

$$\mathbb{E}[X_1+X_2+\cdots+X_n]=\mathbb{E}[X_1]+\mathbb{E}[X_2]+\cdots+\mathbb{E}[X_n]$$

This can be proven by induction.

Indicator Random Variables

For any event A, we can define the indicator random variable $\mathbf{1}[A]$ for A

$$\mathbf{1}[A] = X = \begin{cases} 1 & \text{if event A occurs} \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbb{P}(X = 1) = \mathbb{P}(A)$$
$$\mathbb{P}(X = 0) = 1 - \mathbb{P}(A)$$

$$p_X(k) = \begin{cases} \mathbb{P}(A) & \text{if } k = 1 \\ 1 - \mathbb{P}(A) & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbb{E}[X] = 1 \cdot p_X(1) + 0 \cdot p_X(0) = p_X(1) = \mathbb{P}(A)$$

You'll also see notation like:

$\mathbb{1}[A]$, 1_A, $\mathbb{1}[\text{some boolean}]$

Repeated Coin Tosses (Again)

The probability of flipping a head is p and we want to find the total number of heads flipped when we flip the coin n times?

Let X be the total number of heads

What indicators can we define? What 'Booleans' have enough information to combine (add) and solve the problem?

Repeated Coin Tosses (Again)

The probability of flipping a head is p and we want to find the total number of heads flipped when we flip the coin n times?

Let X be the total number of heads

Define X_i as follows:

$$X_i = \begin{cases} 1 & \text{if the ith coin flip is heads} \\ 0 & \text{otherwise} \end{cases}$$

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The probability of flipping a head is p and we want to find the total number of heads flipped when we flip the coin n times?

$$\mathbb{P}(X_i = 1) = p$$
$$\mathbb{P}(X_i = 0) = 1 - p$$

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$$\mathbb{E}[X_i] = 1 \cdot p + 0 \cdot (1 - p) = p$$

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By **Linearity of Expectation**,

$$\mathbb{E}[X] = \mathbb{E}\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \mathbb{E}[X_i]$$

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$$\mathbb{E}[X] = \mathbb{E}\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \mathbb{E}[X_i] = \sum_{i=1}^{n} p = np$$

Computing complicated expectations

We often use these three steps to solve complicated expectations

1. **Decompose:** Finding the right way to decompose the random variable into sum of simple random variables
$$X = X_1 + X_2 + \cdots + X_n$$
2. **LOE:** Apply Linearity of Expectation
$$\mathbb{E}[X] = \mathbb{E}[X_1] + \mathbb{E}[X_2] + \cdots + \mathbb{E}[X_n]$$
3. **Conquer:** Compute the expectation of each X_i

Often X_i are indicator random variables

Pairs with the same birthday

In a class of m students, on average how many pairs of people have the same birthday?

Decompose: Let X be the number of pairs with the same birthday

LOE:

Conquer:

Pairs with the same birthday

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<u>Decompose:</u> Let X be the number of pairs with the same birthday

Define X_{ij} as follows:

$$X_{ij} = \begin{cases} 1 & \text{if person i, j have the same bithday} \\ 0 & \text{otherwise} \end{cases} \qquad X = \Sigma_{i,j} X_{ij}$$

<u>LOE:</u>

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<u>LOE:</u>

$$\mathbb{E}[X] = \mathbb{E}\left[\Sigma_{i,j} X_{ij}\right] = \Sigma_{i,j} \mathbb{E}\left[X_{ij}\right]$$

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$$\mathbb{E}[X] = \mathbb{E}\left[\Sigma_{i,j} X_{ij}\right] = \Sigma_{i,j} \mathbb{E}\left[X_{ij}\right]$$

<u>Conquer:</u>

$$\mathbb{E}\left[X_{ij}\right] = \mathbb{P}\left(X_{ij} = 1\right) = \frac{365}{365 \cdot 365} = \frac{1}{365}$$

$$\mathbb{E}[X] = \binom{m}{2} \cdot \mathbb{E}\left[X_{ij}\right] = \binom{m}{2} \cdot \frac{1}{365}$$

Rotating the table

n people are sitting around a circular table. There is a name tag in each place. Nobody is sitting in front of their own name tag.

Rotate the table by a random number k of positions between 1 and n-1 (equally likely)

Let X be the number of people that end up in front of their own name tag. **Find $\mathbb{E}[X]$.**

Decompose:

What X_i can we define that have the needed information?

LOE:

Conquer:

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<u>Decompose:</u> Define X_i as follows:

$$X_i = \begin{cases} 1 & \text{if person i sits infront of their own name tag} \\ 0 & \text{otherwise} \end{cases}$$

Note: $X = \Sigma_{i=1}^{n} X_i$

<u>LOE:</u>

$$\mathbb{E}[X] = \mathbb{E}[\Sigma_{i=1}^{n} X_i] = \Sigma_{i=1}^{n} \mathbb{E}[X_i]$$

<u>Conquer:</u>

These X_i are not independent!
That's ok!!

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$$X_i = \begin{cases} 1 & \text{if person i sits infront of their own name tag} \\ 0 & \text{otherwise} \end{cases} \qquad X = \Sigma_{i=1}^{n} X_i$$

LOE:

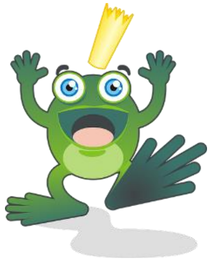
$$\mathbb{E}[X] = \mathbb{E}[\Sigma_{i=1}^{n} X_i] = \Sigma_{i=1}^{n} \mathbb{E}[X_i]$$

Conquer:

$$\mathbb{E}[X_i] = P(X_i = 1) = \frac{1}{n-1} \qquad \mathbb{E}[X] = n \cdot \mathbb{E}[X_i] = \frac{n}{n-1}$$

Extra Practice

Frogger

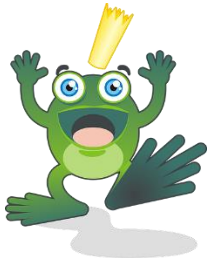


A frog starts on a 1-dimensional number line at 0.

Each second, independently, the frog takes a unit step right with probability p_1, to the left with probability p_2, and doesn't move with probability p_3, where $p_1 + p_2 + p_3 = 1$.

After 2 seconds, let X be the location of the frog. **Find** $\mathbb{E}[X]$.

Frogger – Brute Force



A frog starts on a 1-dimensional number line at 0. At each second, independently, the frog takes a unit step right with probability p_R, to the left with probability p_L, and doesn't move with probability p_S, where $p_L + p_R + p_S = 1$. After 2 seconds, let X be the location of the frog. **Find $\mathbb{E}[X]$.**

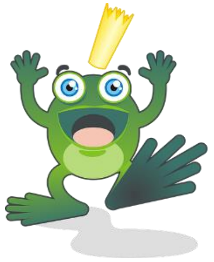
We could find the PMF by computing the probability for each value in the range of X, and then applying definition of expectation:

$$p_X(x) = \begin{cases} p_L^2 & x = -2 \\ 2p_L p_S & x = -1 \\ 2p_L p_R + p_S^2 & x = 0 \\ 2p_R p_S & x = 1 \\ p_R^2 & x = 2 \\ 0 & \text{otherwise} \end{cases}$$

We think about the outcomes that correspond to each value of X and compute the probability of that. For example, X=0 happens when the frog doesn't move – this means it either moved left and then right, or right and then left, or did not move both seconds.

$$\mathbb{E}[\boldsymbol{X}] = \Sigma_\omega P(\omega)X(\omega) = (-2)p_L^2 + (-1)2p_L p_S + 0 \cdot (2p_L p_R + p_S^2) + (1)2p_R p_S + (2)p_R^2 = 2(p_R - p_L)$$

Frogger – LOE



Or we can apply LoE!

A frog starts on a 1-dimensional number line at 0. At each second, independently, the frog takes a unit step right with probability p_R, to the left with probability p_L, and doesn't move with probability p_S, where $p_L + p_R + p_S = 1$. After 2 seconds, let X be the location of the frog. **Find $\mathbb{E}[X]$.**

Define X_i as follows:

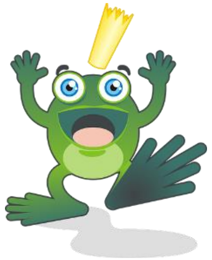
$$X_i = \begin{cases} -1 & \text{if the frog moved left on the } i\text{th step} \\ 0 & \text{otherwise} \\ 1 & \text{if the frog moved right on the } i\text{th step} \end{cases}$$

$$\mathbb{E}[X_i] = -1 \cdot p_L + 1 \cdot p_R + 0 \cdot p_S = (p_R - p_L)$$

By Linearity of Expectation,

$$\mathbb{E}[X] = \mathbb{E}\left[\sum_{i=1}^{2} X_i\right] = \sum_{i=1}^{2} \mathbb{E}[X_i] = 2(p_R - p_L)$$

Frogger – LOE



If we interested in a whole minute (60 sec), the first approach would be awful because we would need to compute many probabilities or deal with a gnarly summation! Instead, we can use LoE!

A frog starts on a 1-dimensional number line at 0. At each second, independently, the frog takes a unit step right with probability p_R, to the left with probability p_L, and doesn't move with probability p_S, where $p_L + p_R + p_S = 1$. After 60 seconds, let X be the location of the frog. **Find** $\mathbb{E}[X]$.

Define X_i as follows:

$$X_i = \begin{cases} -1 & \text{if the frog moved left on the } i\text{th step} \\ 0 & \text{otherwise} \\ 1 & \text{if the frog moved right on the } i\text{th step} \end{cases}$$

$$\mathbb{E}[X_i] = -1 \cdot p_L + 1 \cdot p_R + 0 \cdot p_S = (p_R - p_L)$$

By Linearity of Expectation,

$$\mathbb{E}[X] = \mathbb{E}\left[\sum_{i=1}^{60} X_i\right] = \sum_{i=1}^{60} \mathbb{E}[X_i] = \mathbf{60}(p_R - p_L)$$