Random Variables

CSE 312 24Su

Lecture 8

Outline

Today we will talk about **random variables!**

> Introduce motivation and idea of random variables

> How to describe random variables and their properties

Support, probability mass function, cumulative distribution function, expectation

Defining Events Can Be *Tedious*

For example, if we're interested in analyzing the sum of 2 random dice…

We might define events for all the possible outcomes:

E_1 ~ the sum is 1

E_2 ~ the sum is 2

E_3 ~ the sum is 3

…

E_{12} ~ the sum is 12

Defining and Using Events Can Be *Tedious*

For example, if we're interested in analyzing the sum of 2 random dice…

Now, how might we express the event that the sum is more than 6?

We could define an event $A \sim$ the sum is more than 6 or since that's a bit undescriptive we might look for $P(E_7 \cup E_8 \cup E_9 \cup E_{10} \cup E_{11} \cup E_{12})$

We want a way to easily say something "[the sum of the dice] > 6" or be able to summarize things like "what is [the sum of the dice] on average?"

Random Variables

Random Variable

Informally: A random variable is a way to **summarize** the important (numerical) information from your outcome.

Random Variable

$X : \Omega \to \mathbb{R}$ **is a random variable**

$\boxed{X(\omega)}$ **is the summary of the outcome** ω

Formally: *Function* assigning a value to outcomes of a random experiment

Ω is set of possible outcomes

Outcome $\underbrace{\boldsymbol{\omega \in \Omega}}$ $\rightarrow$ X $\rightarrow$ $\underbrace{\textbf{Quantitative value}}$

The sum of two dice

EVENTS

We could define

$E_2 =$"sum is 2"

$E_3 =$"sum is 3"

…

$E_{12} =$"sum is 12"

And ask "which event occurs"?

RANDOM VARIABLE

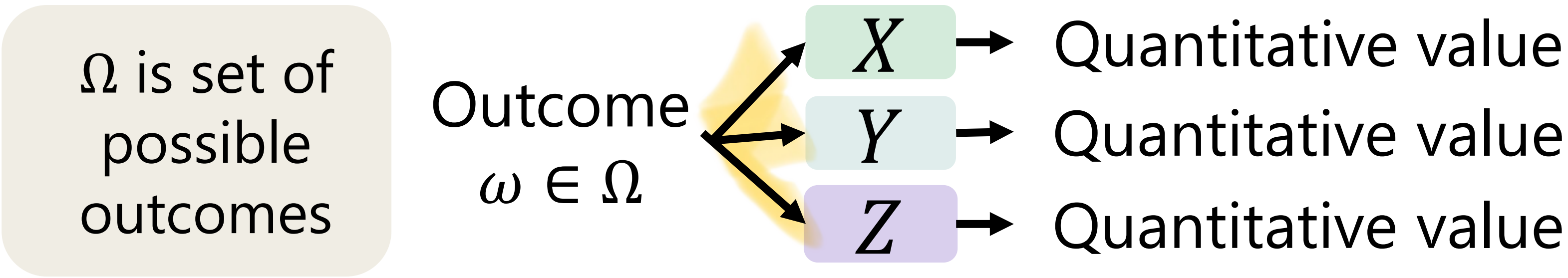
$X: \Omega \rightarrow \mathbb{R}$

X is the sum of the two dice.

$X(4,3) = 7$

More random variables

From one sample space, you can define many random variables.



E.g., Roll a fair red die and a fair blue die, Ω is the set of possible outcomes

Let D be the value of the red die minus the blue die $D(4,2) = 2$

Let S be the sum of the values of the dice $S(4,2) = 6$

Let M be the maximum of the values $M(4,2) = 4$

…

Notational Notes

X

> We will always use capital letters for random variables.

> It's common to use lower-case letters for the values they could take on.

> When we say $X = 2$, we are referring to the set of outcomes that the random variable X assigns the value 2

For example, if X is the number of heads in three coin flips, $X = 2$ corresponds to the set of outcomes $\{HHT, HTH, THH\}$

Notational Notes

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For example, if X is the number of heads in three coin flips, $X = 2$ corresponds to the set of outcomes $\{HHT, HTH, THH\}$

Note that $X = 2$ is a set of outcomes, so it is an event!

How do we describe random variables?

Random variable gives a *quantitative property* of an outcome in a random experiment.

e.g., the number of coin flips till the first head, or the sum of two dice rolls

> **Support**

e.g., what are the possible "number of coin flips" it could possibly take?

> **Probability Mass Function**

e.g., what's the probability it takes 3 coin flips till the 1st head? what about 5 flips?

> **Cumulative Distribution Function**

e.g., what's the probability it takes *less than* 3 coin flips till the 1st head?

> **Expectation**

e.g., how many coin flips can we *expect* it to take till the first head on average?

> **Variance**

e.g., on average, how much does the "number of coin flips" deviate from the expectation?

Support

The "support" (aka "the range") is the set of values X can actually take.

We called this the "image" in 311.

E.g., We roll a red and a blue dice and define these 3 random variables:

D (difference of red and blue dice) has support $\{-5, -4, -3, \ldots, 4,5\}$

$(1,6)$ $(2,6)$ $(1,5)$ $(6,1)$

S (sum) has support $\{2,3, \ldots, 12\}$ $(1,1)$ $(6,6)$

M (max of the two dice) has support $\{1, 2, \ldots 6\}$

Each value in the support corresponds to some outcome(s) from Ω

Support

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We called this the “image” in 311.

We roll a red and a blue dice.

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S (sum) has support $\{2,3, \dots, 12\}$

M (max of the two dice) has support $\{1,2,3,4,5,6\}$

Probability Mass Function

Often we're interested in the event $\{\omega: X(\omega) = k\}$

Which is the event…that $X = k$.

e.g., $S = 7$ is the event that two dice sum to 7

We'll write $\mathbb{P}(X = k)$ to describe the probability of that event

so $\mathbb{P}(S = 2) = \frac{1}{36}$, $\mathbb{P}(S = 7) = \frac{6}{36}$

The function that tells you $\mathbb{P}(X = k)$ is the "probability mass function"

We'll often write $\mathrm{p}_X(k) = \mathbb{P}(X = k)$ for the PMF.

⚠️ A random variable X is **not** an event.

But, $\boldsymbol{X} = \mathbf{2}$ **is an event** – it's the event/set of outcomes where the random variable takes on the value 2

Probability Mass Function

Let T be the number of 2's rolling a (fair) red and blue die.

What is the range of T*?* $\Omega_T = \{0,1,2\}$

What is the PMF of T*?*

6·6

$p_T(0) = 25/36$, $p_T(1) = 10/36$, $p_T(2) = 1/36$

P(T=0) P(T=1)

$$p_T(k) = \begin{cases} 25/36 & k = 0 \\ 10/36 & k = 1 \\ 1/36 & k = 2 \\ 0 & \text{otherwise} \end{cases}$$

> Often use **piecewise function** to give probabilities for different values of k
> This is a function, so include **otherwise case**, so it is defined for all values

Partition

A random variable partitions Ω.

$$\sum_{k \in \Omega_x} p_X(k) = 1$$

Let T be the number of 2's rolling a (fair) red and blue die.

T=0 $p_T(0) = 25/36$

$p_T(1) = 10/36$

$p_T(2) = 1/36$

	D2=1	D2=2	D2=3	D2=4	D2=5	D2=6
D1=1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1.6)
D1=2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
D1=3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
D1=4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
D1=5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
D1=6	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

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A random variable partitions Ω.

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$p_T(2) = 1/36$

	D2=1	D2=2	D2=3	D2=4	D2=5	D2=6
D1=1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1.6)
D1=2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
D1=3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
D1=4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
D1=5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
D1=6	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

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$p_T(2) = 1/36$

	D2=1	D2=2	D2=3	D2=4	D2=5	D2=6
D1=1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1.6)
D1=2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
D1=3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
D1=4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
D1=5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
D1=6	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

Partition

A random variable partitions Ω.

$$\sum_{k \in \Omega_x} p_X(k) = 1$$

Let T be the number of 2's rolling a (fair) red and blue die.

T=1 $p_T(0) = 25/36$

T=2 $p_T(1) = 10/36$

T=7 $p_T(2) = 1/36$

	D2=1	D2=2	D2=3	D2=4	D2=5	D2=6
D1=1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1.6)
D1=2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
D1=3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
D1=4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
D1=5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
D1=6	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

Try It Yourself

There are 20 balls, numbered 1,2,…,20 in an urn.

You'll draw out a size-three subset. (i.e. without replacement)

$\Omega = \{$size three subsets of $\{1, \ldots, 20\}$ $\}$, $\mathbb{P}()$ is uniform measure.

Let X be the largest value among the three balls.
e.g., if the outcome is $\{4,2,10\}$ then $X = 10$.

> What is the **support of $\boldsymbol{X}$**?

> Write down the **PMF of $\boldsymbol{X}$**
i.e., what is $p_X(k) = \mathbb{P}(X = k)$?

Fill out the poll everywhere so Claris knows how long to explain
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Try It Yourself $(1,2,3)$

There are 20 balls, numbered 1,2,…,20 in an urn.

You'll draw out a size-three subset. (i.e. without replacement)

$\Omega = \{$size three subsets of $\{1, \dots, 20\}$ $\}$, $\mathbb{P}()$ is uniform measure.

Let X be the largest value among the three balls.

$\Omega_X = \{3, 4, 5, \dots, 20\}$

$$p_X(k) = \begin{cases} \dfrac{\binom{k-1}{2}}{\binom{20}{3}} & k \in \{3, 4, 5, \dots, 20\} \\ 0 & \text{otherwise} \end{cases}$$

$P(X=k)$

Try It Yourself

There are 20 balls, numbered 1,2,…,20 in an urn.

You'll draw out a size-three subset. (i.e. without replacement)

Let X be the largest value among the three balls.

$$\Omega_X = \{3,4,5,\ldots,19,20\}$$

$$p_X(k) = \begin{cases} \binom{k-1}{2} / \binom{20}{3} & \text{if } k \in \mathbb{N},\ 3 \le k \le 20 \\ 0 & \text{otherwise} \end{cases}$$

Good checks:
if you sum up $p_X(k)$ do you get 1?
is $p_X(k) \geq 0$ for all k? Is it defined for all k?

Cumulative Distribution Function (CDF)

The **PMF gives the probability** $\boldsymbol{X = k}$
(and is the most common way to describe a random variable)

There's a second representation:

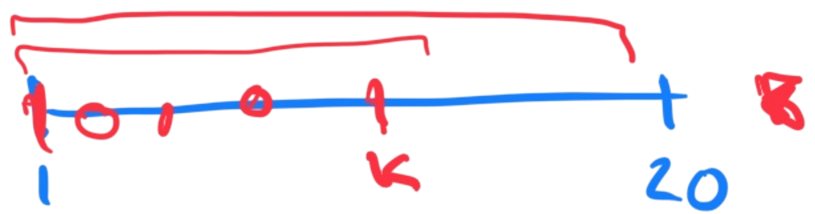
The **cumulative distribution function (CDF) gives the probability** $\boldsymbol{X \leq k}$

More formally, $\mathbb{P}(\{\omega : X(\omega) \leq k\})$

Often written $\boldsymbol{F_X(k)} = \mathbb{P}(\boldsymbol{X \leq k})$

$F_X(k) = \sum_{i : i \leq k} p_X(i)$
"sum up the probabilities of X taking all possible numbers less than k"

Try it yourself



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$\Omega = \{$size three subsets of $\{1, \dots, 20\}$ $\}$, $\mathbb{P}()$ is uniform measure.

Let X be the largest value among the three balls.

$\Omega_X = \{3,4,5, \dots, 19, 20\}$

$\lfloor\ \rfloor$

$$F_X(k) = \mathbb{P}(X \le k) = \begin{cases} & k < 3 \\ \dfrac{\binom{\lfloor k \rfloor}{3}}{\binom{20}{3}} & 3 \le k \le 20 \\ & k > 20 \end{cases}$$

10

105

Think "what is the probability the largest value is less than or equal to 10?"

Try it yourself

There are 20 balls, numbered 1,2,…,20 in an urn.

You'll draw out a size-three subset. (i.e. without replacement)

$\Omega = \{$size three subsets of $\{1, \dots, 20\}$ $\}$, $\mathbb{P}()$ is uniform measure.

Let X be the largest value among the three balls.

$\Omega_X = \{3,4,5, \dots, 19, 20\}$

$$F_X(k) = \mathbb{P}(X \le k) = \begin{cases} 0 & \text{if } k < 3 \\ \binom{\lfloor k \rfloor}{3} / \binom{20}{3} & \text{if } 3 \le k \le 20 \\ 1 & \text{otherwise} \end{cases}$$

k>20

Think "what is the probability the largest value is less than or equal to k?"

Try it yourself

There are 20 balls, numbered 1,2,…,20 in an urn.

You'll draw out a size-three subset. (i.e. without replacement)

Let X be the largest value among the three balls.

$$\Omega_X = \{3,4,5,\dots,19,20\}$$

$$F_X(k) = \mathbb{P}(X \leq k) = \begin{cases} 0 & \text{if } k < 3 \\ \binom{\lfloor k \rfloor}{3} / \binom{20}{3} & \text{if } 3 \leq k \leq 20 \\ 1 & \text{otherwise} \end{cases}$$

$k > 20$

Good checks:

Is $F_X(\infty) = \mathbb{P}(X \leq \infty) = 1$? If not, something is wrong.

Is $F_X(k)$ increasing? If not something is wrong.

Is $F_X(k)$ defined for all real number inputs? If not something is wrong.

Summary (PMF and CDF)

PROBABILITY MASS FUNCTION

Defined for all $\mathbb{R}$ inputs.

Usually has "0 otherwise" as an extra case.

$\sum_x p_X(x) = 1$

$0 \leq p_X(x) \leq 1$

$\sum_{i:i \leq k} p_X(k) = F_X(k)$

CUMULATIVE DISTRIBUTION FUNCTION

Defined for all $\mathbb{R}$ inputs.

Often has "0 otherwise" and 1 otherwise" extra cases

Non-decreasing function

$0 \leq F_X(k) \leq 1$

$\lim_{x \to -\infty} F_X(k) = 0$

$\lim_{x \to \infty} F_X(k) = 1$

Expectation

Expectation

Expectation

The "expectation" (or "expected value") of a random variable X is:

$$\mathbb{E}[X] = \sum_{k \in \Omega_X} k \cdot \underbrace{\mathbb{P}(X = k)}_{\text{PMF } p_X(k)}$$

Intuition: The **weighted average of values X could take on**.

Weighted by the probability you actually see them.

e.g., if $Y \sim$ num. flips to get a head,

$E[Y]$ is num. of flips we expect it to take *on average*

Example 1

Flip a fair coin twice (independently).

What is the *expected number of heads* we see?

Let X be the number of heads. $E[X] = ?$

$\Omega = \{HH, HT, TH, TT\}$

$\Omega_X = \{0, 1, 2\}$

$$p_X(k) = \begin{cases} 1/4 & k=0 \\ 2/4 & k=1 \\ 1/4 & k=2 \\ 0 & \text{o.w.} \end{cases}$$

$E[X] = 0 \cdot \frac{1}{4} + 1 \cdot \frac{2}{4} + 2 \cdot \frac{1}{4}$

$= 1$

Example 1

Flip a fair coin twice (independently).
What is the *expected number of heads* we see?

Let X be the number of heads.

$\Omega = \{TT, TH, HT, HH\}$, $\mathbb{P}()$ is uniform measure.

$\Omega_X = \{0,1,2\}$

$$\mathbb{E}[X] = p_X(0) \cdot 0 + p_X(1) \cdot 1 + p_X(2) \cdot 2$$
$$= \frac{1}{4} \cdot 0 + \frac{1}{2} \cdot 1 + \frac{1}{4} \cdot 2 = 0 + \frac{1}{2} + \frac{1}{2} = 1.$$

Example 2

You roll a biased die.

It shows a 6 with probability $\frac{1}{3}$, and 1,…,5 with probability $2/15$ each.

Let X be the value of the die. What is $\mathbb{E}[X]$?

1. Write the PMF for X: $p_X(k) = \begin{cases} 2/15 & k \in \{1,2,3,4,5\} \\ 1/3 & k = 6 \\ 0 & \text{otherwise} \end{cases}$

2. Plug in formula: $\mathbb{E}[X] = \frac{2}{15} \cdot 1 + \frac{2}{15} \cdot 2 + \frac{2}{15} \cdot 3 + \frac{2}{15} \cdot 4 + \frac{2}{15} \cdot 5 + \frac{1}{3} \cdot 6 = 4$

$\mathbb{E}[X]$ is not just the most likely outcome!

Try it yourself

> Let X be the result shown on a fair die. What is $\mathbb{E}[X]$?

> Let Y be the sum of two (independent) fair die rolls. What is $\mathbb{E}[Y]$?

Fill out the poll everywhere so Claris knows how long to explain
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Try it yourself

Let X be the result shown on a fair die. What is $\mathbb{E}[X]$

1. Write the PMF: $p_X(k) = \begin{cases} 1/6 & k \in \{1,2,3,4,5,6\} \\ 0 & \text{otherwise} \end{cases}$

2. Plug into formula: $\mathbb{E}[X] = 6 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 1 \cdot \frac{1}{6}$

$$= \frac{21}{6} = 3.5$$

$\mathbb{E}[X]$ is not necessarily a possible outcome!

That's ok, it's an average!

Try it yourself

Let Y be the sum of two (independent) fair die rolls. What is $\mathbb{E}[Y]$?

$\mathbb{E}[Y] =$

$$\frac{1}{36} \cdot 2 + \frac{2}{36} \cdot 3 + \frac{3}{36} \cdot 4 + \frac{4}{36} \cdot 5 + \frac{5}{36} \cdot 6 + \frac{6}{36} \cdot 7 + \frac{5}{36} \cdot 8 + \frac{4}{36} \cdot 9 + \frac{3}{36} \cdot 10 + \frac{2}{36} \cdot 11 + \frac{1}{36} \cdot 12$$

$= 7$

$\mathbb{E}[Y] = 2\mathbb{E}[X]$. That's not a coincidence…we'll talk about why next time!

Subtle but Important

X is random. You don't know what it is (at least until you run the experiment).

$\mathbb{E}[X]$ is not random. It's a number.

You don't need to run the experiment to know what it is.

Summary

Today we have talked about **random variables** which assign quantitative values to outcomes of a random experiment.

How to describe a random variable?

> **Range/Support:** $\mathbf{\Omega_X}$ is set of possible values X can be

> **Probability Mass Function:** $p_X(k) = \mathbb{P}(X = k)$

> **Cumulative Distribution Function:** $F_X(k) = \mathbb{P}(X \leq k)$

> **Expectation**: A single value that is the weighted average of values on the support, weighted on the probabilities X takes on each

> **Variance:** Coming soon!

You have 10 pairs of socks (so 20 socks in total), with each pair being a different color. You put them in the washing machine, but the washing machine eats 4 of the socks chosen at random. Every subset of 4 socks is equally probable to be the subset that gets eaten. Let X be the number of complete pairs of socks that you have left.

More Practice: Infinite sequential processes

Infinite sequential process

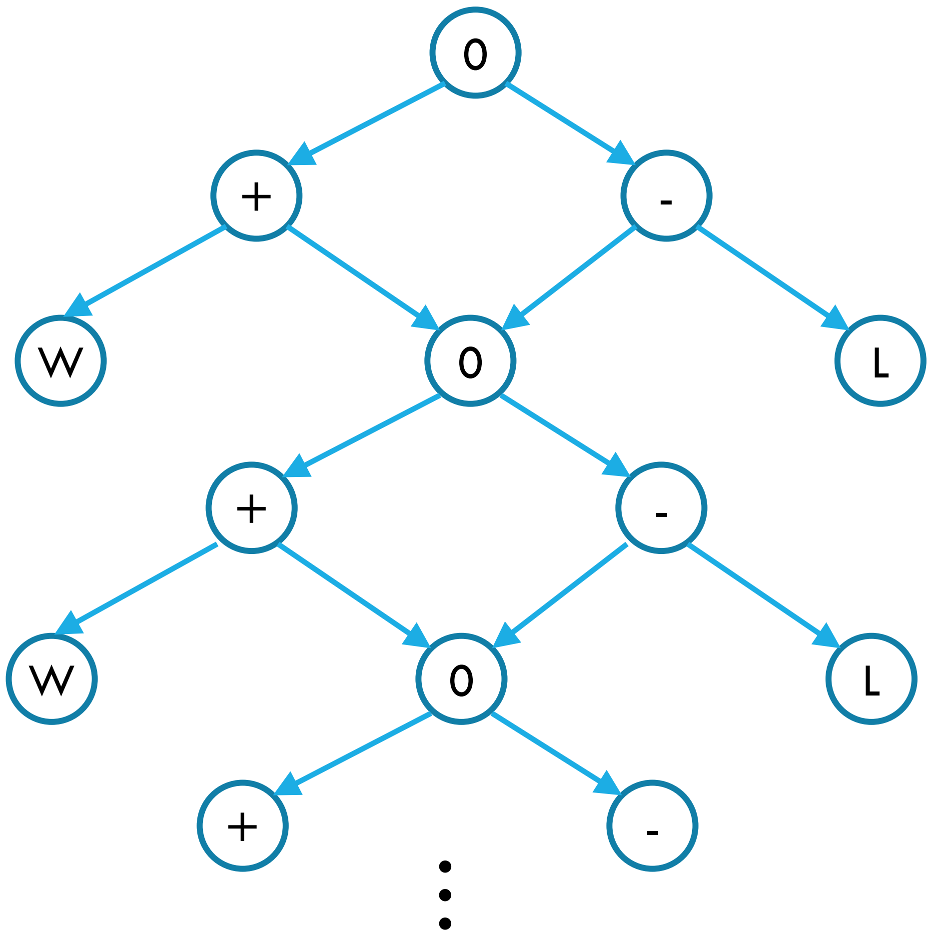
In volleyball, sets are played first team to

- Score 25 points
- Lead by at least 2

At the same time wins a set.

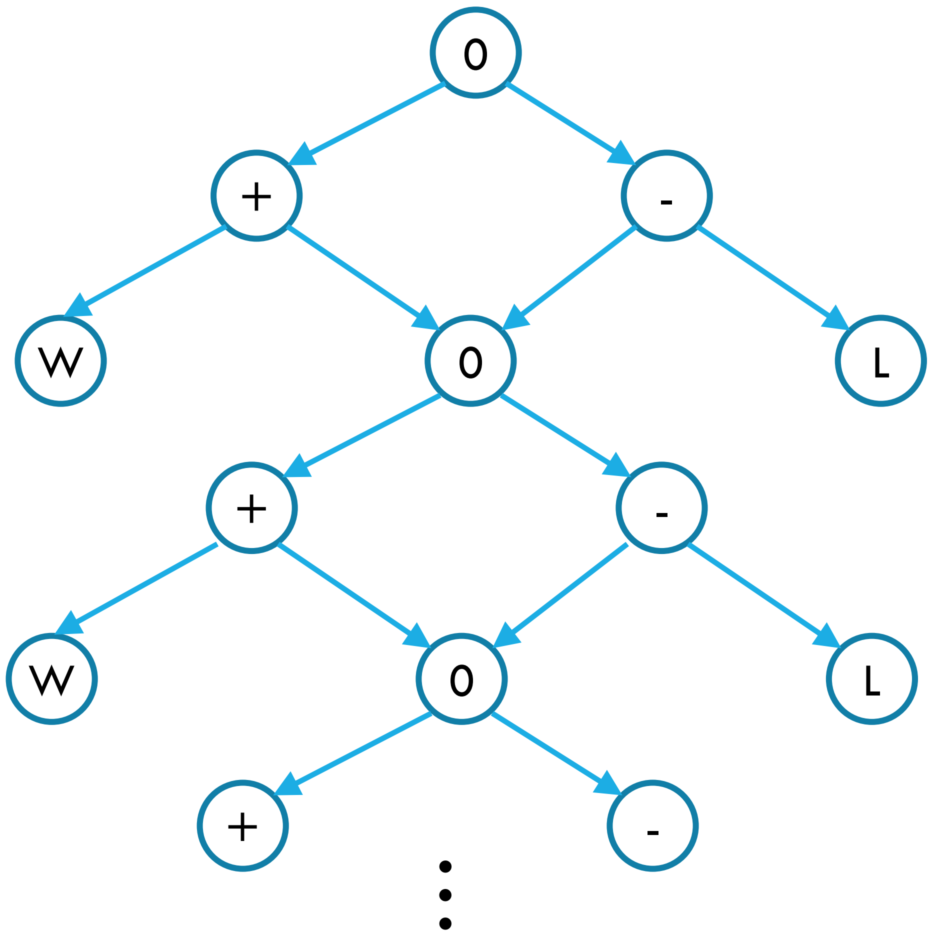
Suppose a set is 23-23. Your team wins each point independently with probability p. What is the probability your team wins the set?

Sequential Process



$$\mathbb{P}(win\ from\ even) = p^2 + 2p(1-p)\mathbb{P}(win\ from\ even)$$

Sequential Process



$$\mathbb{P}(win\ from\ even) = p^2 + 2p(1-p)\mathbb{P}(win\ from\ even)$$

$$x - x[2p - p^2] = p^2$$
$$x[1 - 2p + p^2] = p^2$$

$$x = \frac{p^2}{p^2 - 2p + 1}$$

More Random Variable Practice

Roll a fair die n times. Let Z be the number of rolls that are $5s$ or $6s$.

What is the pmf?

Don’t try to write the CDF…it’s a mess…

Or try for a few minutes to realize it isn’t nice.

More Random Variable Practice

Roll a fair die n times. Let Z be the number of rolls that are $5s$ or $6s$.

What’s the probability of getting exactly k 5’s/6’s? Well we need to know which k of the n rolls are 5’s/6’s. And then multiply by the probability of getting exactly that outcome

$$p_Z(z) = \begin{cases} \binom{n}{z} \cdot \left(\frac{1}{3}\right)^z \left(\frac{2}{3}\right)^{n-z} & \text{if } z \in Z, 0 \leq z \leq n \\ 0 & \text{otherwise} \end{cases}$$