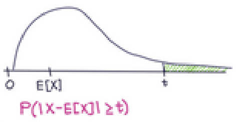
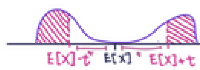
tail bounds

LECTURE 18

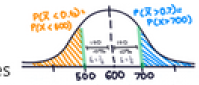
- Markov's inequality - $\mathbb{P}(X \geq t) \leq \frac{\mathbb{E}[X]}{t}$
  - Use if X is non-negative and we know the expectation
  - Useful when we don't know much about X



- Chebyshev's inequality - $\mathbb{P}(|X - \mathbb{E}[X]| \geq t) \leq \frac{\text{Var}(t)}{t^2}$
  - Use if we know the expectation **and** variance of X
  - Gives better bounds with small variances



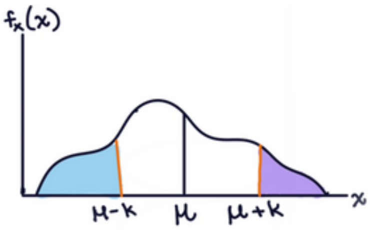
- Chernoff Bound
  $\mathbb{P}(X \leq (1-\delta)\mu) \leq e^{\left(-\frac{\delta^2\mu}{2}\right)}$ *and* $\mathbb{P}(X \geq (1+\delta)\mu) \leq e^{\left(-\frac{\delta^2\mu}{3}\right)}$
  - Use if X is a sum of independent Bernoulli random variables
  - Gives a very good bound usually, and is especially helpful when X is binomial and we can't easily computationally compute some summations/probability



- Union Bound - $\mathbb{P}(A \cup B) \leq \mathbb{P}(A) + \mathbb{P}(B)$
  - Use if we don't have enough information to find the union (e.g,. ways for at least of __ to occur, for A, or B, or C, or ... to occur)

WHAT ARE TAIL BOUNDS?

A tail bound (or concentration inequality) is a statement bounds the probability in the "tails" of the distribution (e.g., there's little probability far from the center) or (equivalently) the probability is concentrated near the expectation.



MARKOV'S INEQUALITY

Let X be a random variable supported (only) on non-negative numbers. For any $t > 0$

$$\mathbb{P}(X \geq t) \leq \frac{\mathbb{E}[X]}{t}$$

Requirements:

1. X must be non-negative
2. We know the expectation of X

Let X be a random variable supported (only) on non-negative numbers. For any $k > 0$

$$\mathbb{P}(X \geq k\mathbb{E}[X]) \leq \frac{1}{k}$$

you'll rarely use this form

Why is this true?

You roll a fair die until you see a 6. Let X be the number of rolls. Bound P(X≥12.)

The average number of ads is 25. Upper bound the prob. >= 75 ads.

The average number of ads is 25. Upper bound the prob. >= 20 ads.

Sometimes bounds give us unhelpful results. e.g., P(X>=20)<=1.25 is not helpful, we already knew that

CHEBYSHEV'S INEQUALITY

Let X be a random variable. For any $t > 0$

$$\mathbb{P}(|X - \mathbb{E}[X]| \geq t) \leq \frac{\text{Var}(X)}{t^2}$$

Requirements:

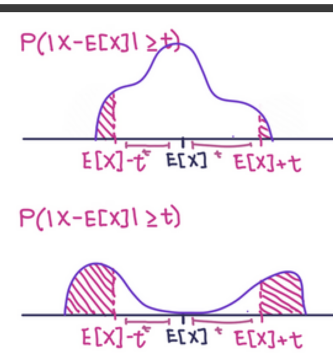
1. We know the *expectation* of X
2. We know the *variance* of Xc

Let X be a random variable. For any $k > 0$

$$\mathbb{P}\left(|X - \mathbb{E}[X]| \geq k\sqrt{\text{Var}(X)}\right) \leq \frac{1}{k^2}$$

you'll rarely use this form

Why is this true?



You roll a fair die until you see a 6. Let X be the number of rolls. Bound P(X≥12.)

The average number of ads is 25. The variance is 16. Upper bound the prob. >= 75 ads.

*You run a poll of 1000 people where in the true population 60% of population supports you. What is the probability that the poll is **not** within 10% of the true value?*

CHERNOFF BOUND

Let $X_1, X_2, \dots, X_n$ be ***independent*** Bernoulli random variables.
Let $X = \sum X_i$, and $\mu = \mathbb{E}[X]$. For any $0 \le \delta \le 1$

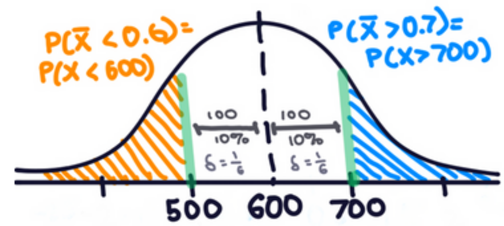
$\mathbb{P}(X \le (1-\delta)\mu) \le e^{\left(-\frac{\delta^2 \mu}{2}\right)}$ *and* $\mathbb{P}(X \ge (1+\delta)\mu) \le e^{\left(-\frac{\delta^2 \mu}{3}\right)}$

LEFT TAIL — RIGHT TAIL

Requirements:
1. X is a sum of independent Bernoulli random variables.
2. We know $\mathbb{E}[X]$

*You run a poll of 1000 people where 60% of true population supports you. What is the probability that the poll is **not** within 10% of the true value?*

Goal: bound $\mathbb{P}(|\bar{X} - 0.6| \ge 0.1) = \mathbb{P}(\bar{X} \le 0.5) + \mathbb{P}(\bar{X} \ge 0.7)$



1. bound the left tail

$\mathbb{P}(\bar{X} \le 0.5)$

2. bound the right tail

$\mathbb{P}(\bar{X} \ge 0.7)$

3. putting it together

UNION BOUND

For any events E, F
$\mathbb{P}(E \cup F) \le \mathbb{P}(E) + \mathbb{P}(F)$

Sometimes we don't' have enough information to compute this probability exactly, so we use the union bound to bound that probability

There are 20 frogs on each location in a 5x5 grid. Each frog independently jumps to the L, R, U, D, or neither with equal probability. A frog at an edge of the grid magically warps to the corresponding edge (ignore "edge" cases). Bound the probability at least one square ends up with at least 36 frogs.

1. Apply Union Bound

2. Apply Chernoff Bound to bound each of P(Ai)

3. Put it all together