More CLT + Joint Probability

CSE 312 24Su

Lecture 16

Announcements

HW5 released tonight

Midterm grades and solutions released

"Where grades stand" post later this week

- Feel free to set up a short meeting with me to discuss concerns.
- Useful if you're worried about hitting particular targets.
- Also useful if your midterm grade was very different from your homework scores.

Today

- Central Limit Theorem (CLT) – one more use case!
- Analyzing Relationships Between Multiple Random Variables
  - Joint Distributions (joint support, joint PMF, etc.)
  - Covariance

Central Limit Theorem (Review)

"The sum of **any** independent random variables **approaches** a normal distribution. It becomes closer to normal/the approximation gets better as we sum more RVs together."

Central Limit Theorem

If $X_1, X_2, \ldots, X_n$ are i.i.d. random variables, each with mean μ and variance σ^2

Let $Y_n = X_1 + X_2 + \cdots + X_n$

As $n \to \infty$, Y_n approaches a normal distribution $\mathcal{N}(n \cdot \mu, n \cdot \sigma^2)$

(i.e., CDF of Y_n converges to the CDF of $\mathcal{N}(n \cdot \mu, n \cdot \sigma^2)$)

Outline of CLT steps

1. Setup the problem (e.g., $X = \sum_{i=1}^{n} X_i$, X_i are i.i.d., and we want $\mathbb{P}(X \leq k)$)
   Write event you are interested in, in terms of sum of random variables.

   ⭐ Apply *continuity correction here* if RVs are discrete.

2. Apply CLT (e.g., approx X as $Y \sim \mathcal{N}(n\mu, n\sigma^2)$ -> $\mathbb{P}(X \leq k) \approx \mathbb{P}(Y \leq k)$)
   Approximate sum of RVs as normal with appropriate mean and variance

from here, we're working with a normal distribution, which we've worked with before!

3. Compute probability approximation using Phi table
   > *Standardize* $(Z = \frac{N-\mu}{\sigma})$ -> $\mathbb{P}(Y \leq k) = \mathbb{P}\left(\frac{Y-\mu}{\sigma} \leq \frac{k-\mu}{\sigma}\right) = \mathbb{P}\left(Z \leq \frac{k-\mu}{\sigma}\right)$

   > *Write in terms of* $\Phi(z) = \mathbb{P}(Z \leq z)$

   > *Look up in table*

What if we are asked something like…

> How people need to be surveyed to draw __ conclusion about *[something to do with a sum of i.i.d* RVs' with __ probability?

> How many trials should we do till the average amount from the trails [*sum of i.i.d RVs*] is __ away from the mean with __ probability?

> What should the standard deviation be in order for the probability of [*something to do with a sum of i.id. RVs]* to be ___?

We will follow the exact same process as before for using CLT! Except now, we'll end up with something like $\mathbb{P}(Y \leq c) = 0.96$ where we need to solve for c

Cereal

A cereal company claims that their boxes contain an average of 500 grams of cereal with a variance of 100 grams. To test this claim, you take a random sample of n boxes. You want to determine the sample size n such that the probability of the sample mean being within 2 grams of the true mean (i.e., between 498 grams and 502 grams) is at least 95%.

1. Setup the Problem.

$\left(\frac{X_i}{n}\right)$

2. Apply CLT

3. Compute Probability

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1. Setup the Problem.

Let X_i be the weight of the ith box weighed. The sample mean is $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i = \sum_{i=1}^{n}(\frac{X_i}{n})$

Goal: $\mathbb{P}(498 \leq \bar{X} \leq 502) \geq 0.95$

2. Apply CLT

3. Compute Probability

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Goal: $\mathbb{P}(498 \leq \bar{X} \leq 502) \geq 0.95$

2. Apply CLT

$\bar{X}$ is a sum of i.i.d random variables each with $\mu = E\left[\frac{X_i}{n}\right] = \frac{500}{n}$ and $\sigma^2 = Var\left(\frac{X_i}{n}\right) = \frac{100}{n^2}$

$\frac{1}{n}E[X_i]$ $\frac{1}{n^2}Var(X_i)$

Let $Y \sim \mathcal{N}\left(n \cdot \frac{500}{n}, n \cdot \frac{100}{n^2}\right) = \mathcal{N}\left(500, \frac{100}{n}\right)$ → <u>By CLT</u>, $\mathbb{P}(498 \leq \bar{X} \leq 502) \approx \mathbb{P}(498 \leq Y \leq 502)$

3. Compute Probability

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$\mathbb{P}(498 \leq Y \leq 502) = \mathbb{P}\left(\frac{498-500}{\sqrt{100/n}} \leq Z \leq \frac{502-500}{\sqrt{100/n}}\right) = \mathbb{P}\left(\frac{-2}{10/\sqrt{n}} \leq Z \leq \frac{2}{10/\sqrt{n}}\right)$ *standardize*

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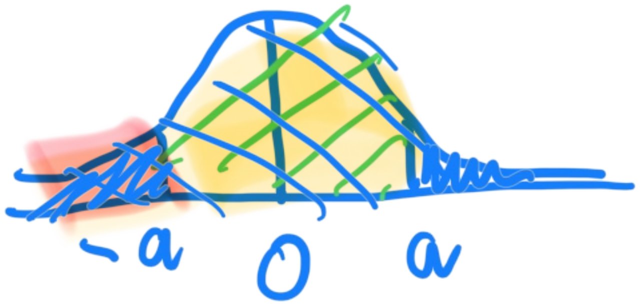
Let $Y \sim \mathcal{N}\left(n \cdot \frac{500}{n}, n \cdot \frac{100}{n^2}\right) = \mathcal{N}\left(500, \frac{100}{n}\right) \rightarrow$ By CLT, $\mathbb{P}(498 \le \bar{X} \le 502) \approx \mathbb{P}(498 \le Y \le 502)$

3. Compute Probability

$$\mathbb{P}(498 \le Y \le 502) = \mathbb{P}\left(\frac{498-500}{\sqrt{100/n}} \le Z \le \frac{502-500}{\sqrt{100/n}}\right) = \mathbb{P}\left(\underbrace{\frac{-2}{10/\sqrt{n}}}_{-a} \le Z \le \underbrace{\frac{2}{10/\sqrt{n}}}_{a}\right)$$ *standardize*



$$= P(Z \le a) - P(Z \le -a)$$

$$= P(Z \le a) - (1 - P(Z \le a))$$

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$$= \Phi\left(\frac{2}{10/\sqrt{n}}\right) - \left(1 - \Phi\left(\frac{2}{10/\sqrt{n}}\right)\right) = 2\Phi\left(\frac{2}{10/\sqrt{n}}\right) - 1$$ *write in terms of* Φ

$$\geq 0.95$$

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$$= \Phi\left(\frac{2}{10/\sqrt{n}}\right) - \left(1 - \Phi\left(\frac{2}{10/\sqrt{n}}\right)\right) = 2\Phi\left(\frac{2}{10/\sqrt{n}}\right) - 1$$ *write in terms of* $\mathbf{\Phi}$

$$\ge \mathbf{0.95}$$

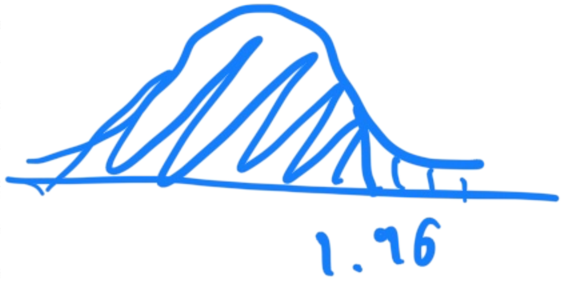
~~Doing some algebra...~~

$2\Phi\left(\frac{2}{10/\sqrt{n}}\right) - 1 \ge 0.95$ *with some algebra, we get* $\Phi\left(\frac{2}{10/\sqrt{n}}\right) \ge 0.975$

reverse table lookup: what can we plug into the z-table to get $\Phi(z) \ge 0.975$?

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.5279	0.53188	0.53586
0.1	0.53983	0.5438	0.54776	0.55172	0.55567	0.55962	0.56356	0.56749	0.57142	0.57535
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.6293	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.6591	0.66276	0.6664	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.7054	0.70884	0.71226	0.71566	0.71904	0.7224
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.7549
0.7	0.75804	0.76115	0.76424	0.7673	0.77035	0.77337	0.77637	0.77935	0.7823	0.78524
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1.0	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.8665	0.86864	0.87076	0.87286	0.87493	0.87698	0.879	0.881	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.9032	0.9049	0.90658	0.90824	0.90988	0.91149	0.91309	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.9222	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
1.6	0.9452	0.9463	0.94738	0.94845	0.9495	0.95053	0.95154	0.95254	0.95352	0.95449
1.7	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.9608	0.96164	0.96246	0.96327
1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
1.9	0.97128	0.97193	0.97257	0.9732	0.97381	0.97441	0.975	0.97558	0.97615	0.9767
2.0	0.97725	0.97778	0.97831	0.97882	0.97932	0.97982	0.9803	0.98077	0.98124	0.98169
2.1	0.98214	0.98257	0.983	0.98341	0.98382	0.98422	0.98461	0.985	0.98537	0.98574
2.2	0.9861	0.98645	0.98679	0.98713	0.98745	0.98778	0.98809	0.9884	0.9887	0.98899
2.3	0.98928	0.98956	0.98983	0.9901	0.99036	0.99061	0.99086	0.99111	0.99134	0.99158
2.4	0.9918	0.99202	0.99224	0.99245	0.99266	0.99286	0.99305	0.99324	0.99343	0.99361
2.5	0.99379	0.99396	0.99413	0.9943	0.99446	0.99461	0.99477	0.99492	0.99506	0.9952
2.6	0.99534	0.99547	0.9956	0.99573	0.99585	0.99598	0.99609	0.99621	0.99632	0.99643
2.7	0.99653	0.99664	0.99674	0.99683	0.99693	0.99702	0.99711	0.9972	0.99728	0.99736
2.8	0.99744	0.99752	0.9976	0.99767	0.99774	0.99781	0.99788	0.99795	0.99801	0.99807
2.9	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861
3.0	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.999

$\Phi(1.96) \approx 0.975$



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3. Compute Probability

$$\mathbb{P}(\mathbf{498} \leq \boldsymbol{Y} \leq \mathbf{502}) = \mathbb{P}\left(\frac{498-500}{\sqrt{100/n}} \leq Z \leq \frac{502-500}{\sqrt{100/n}}\right) = \mathbb{P}\left(\frac{-2}{10/\sqrt{n}} \leq Z \leq \frac{2}{10/\sqrt{n}}\right)$$ *standardize*

$$= \Phi\left(\frac{2}{10/\sqrt{n}}\right) - \left(1 - \Phi\left(\frac{2}{10/\sqrt{n}}\right)\right) = 2\Phi\left(\frac{2}{10/\sqrt{n}}\right) - 1$$ *write in terms of* $\boldsymbol{\Phi}$

$$\geq \mathbf{0.95}$$

Doing some algebra...

$2\Phi\left(\frac{2}{10/\sqrt{n}}\right) - 1 \geq 0.95$ *with some algebra, we get* $\Phi\left(\frac{2}{10/\sqrt{n}}\right) \geq 0.975$

reverse table lookup: what can we plug into the z-table to get $\Phi(z) \geq 0.975$?

$\frac{2}{10/\sqrt{n}} \geq 1.96$ -> *solve the inequality* -> $n \geq 97$

Some more practice with that reverse z-table lookup step!

> What value of c such that $\Phi(c) = 0.76$?

> What condition on c so that $\Phi(c) \geq 0.94$?

> What condition on c so that $\Phi(c) \leq ick\ 0.94$?

once we have that expression, we solve for whatever we're interested in c!

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2.9	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861
3.0	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.999

Some more practice with that reverse z-table lookup step!

> What value of c such that $\Phi(c) = 0.76$?

$c \approx 0.53$ (it's an estimate, it gives us a value that's pretty close to 0.76)

some write this as $\boldsymbol{\Phi^{-1}(0.76) \approx 0.53}$

> What condition on c so that $\Phi(c) \geq 0.94$?

$\Phi(1.56) = 0.94062$. 1.56 is the first value that gives us a value ≥ 0.94

So, if $\boldsymbol{c \geq 1.56}$, $\Phi(c) \geq 0.94$

> What condition on c so that $\Phi(c) \leq 0.94$?

Now, if we want the probability ≤ 0.94, $\boldsymbol{c \leq 1.55}$ because $\Phi(1.55) = 0.93943$

if we said $c \leq 1.56$ *that would include* $\Phi(1.56) = 0.94062 > 0.94$ *and values that give probability between* 0.94 *and* 0.94062

once we have that expression, we solve for whatever we're interested in c!

When to use CLT

Use the CLT when:

- The random variable you're interested in is the sum of independent random variables.
- The random variable you're interested in does not have an easily accessible or easy to use pmf/pdf (or the question you're asking doesn't lend it self to easily using the pmf/pdf)
- You only need an approximate answer, and the sum is of at least a moderate number of random variables.

Relationships between *Multiple* Random Variables

Analyzing *Multiple* Random Variables

So far, we've pretty much only analyzed one random variable at a time

We've *worked* with multiple random variables (e.g., the sum of them) but haven't really analyzed their relationships except for independence

Today, we will talk about analyzing the relations between multiple RVs

> Joint Distributions

Joint PMF/PDF, Joint CDF, Joint Expectation

> Covariance

Quantitative property measuring relationship/dependence of the two RVs

Analyzing *Multiple* Random Variables

Examples:

- Economics: Analyzing the relationship between stock returns and volume helps in understanding market behavior and making investment decisions
- Healthcare: Understanding how different factors (like blood pressure and cholesterol levels) affects the probability of medical conditionals
- Machine learning: understanding how different features contribute to predicting a label and improving model accuracy (what features are most important)
- Recommendation systems: Understanding relationship between user preferences and item features to improve recommendation systems, or other factors like age and choices of products

Joint Distributions

We'll start with the **discrete** case

We have two **discrete** random variables X and Y

(that may or may not be independent)

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Joint **Support/Range** - $\Omega_{X,Y}$

$\Omega_{X,Y}$ is the set of all possible **pairs** of values X and Y can be together

> $\Omega_{X,Y} = \{(a,b) : p_{X,Y}(a,b) > 0\} \subseteq \Omega_X \times \Omega_Y$

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> $\Omega_{X,Y} = \{(a,b) : p_{X,Y}(a,b) > 0\} \subseteq \Omega_X \times \Omega_Y$

Joint **PMF** *(probability mass function)* - $p_{X,Y}(a,b)$

$p_{X,Y}(a,b)$ defines probabilities of X and Y being certain values at once

> $p_{X,Y}(a,b) = \mathbb{P}(X = a \cap Y = b) = \mathbb{P}(X = a, Y = b)$

*should be defined for **all** values of a and b*

Normalization Property: $\sum_{(a,b)\in\Omega_{X,Y}} p_{X,Y}(a,b) = 1$

We have two **discrete** random variables X and Y
(that may or may not be independent)

Joint **Support/Range** - $\Omega_{X,Y}$

$\Omega_{X,Y}$ is the set of all possible **pairs** of values X and Y can be together

> $\Omega_{X,Y} = \{(a, b) : p_{X,Y}(a, b) > 0\} \subseteq \Omega_X \times \Omega_Y$

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> *should be defined for **all** values of a and b*

Joint **CDF** *(cumulative distribution function)* - $F_{X,Y}(a, b)$

> $F_{X,Y}(a, b) = \mathbb{P}(X \leq a \cap Y \leq b) = \mathbb{P}(X \leq a, Y \leq b)$
> *should be defined for **all** values of a and b*

We have two **discrete** random variables X and Y
(that may or may not be independent)

(Joint) **Independence**

X and Y are independent if:

> $p_{X,Y}(a,b) = p_X(a) \cdot p_X(b)$ for all $(a,b) \in \Omega_{X,Y}$

> $\Omega_{X,Y} = \Omega_X \times \Omega_Y$

if you find that $\Omega_{X,Y} \neq \Omega_X \times \Omega_Y$, that is enough to show that they aren't independent

This is the same definition we've seen before for independence of random variables

$\mathbb{P}(X = a, Y = b) = \mathbb{P}(X = a) \cdot \mathbb{P}(X = b)$

Example: Weird Dice

Roll 2 fair 4-sided dice independently

Let X be the value of the 1st dice
Let Y be the value of the 2nd dice

$\Omega_X = \Omega_Y = \{1,2,3,4\}$

The **joint support** is $\Omega_{X,Y} = \Omega_X \times \Omega_Y$

because all combinations are possible

What is the **joint PMF**?

$p_{X,Y}(a,b) =$

$p_{X,Y}$	Y=1	Y=2	Y=3	Y=4
X=1				
X=2				
X=3				
X=4				

Example: Weird Dice

Roll 2 fair 4-sided dice independently

Let X be the value of the 1st dice
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The **joint support** is $\Omega_{X,Y} = \Omega_X \times \Omega_Y$
because all combinations are possible

What is the **joint PMF?**

$$p_{X,Y}(a,b) = \begin{cases} 1/16 & \text{if } a,b \in \Omega_{X,Y} \\ 0 & \text{otherwise} \end{cases}$$

$p_{X,Y}$	Y=1	Y=2	Y=3	Y=4
X=1	1/16	1/16	1/16	1/16
X=2	1/16	1/16	1/16	1/16
X=3	1/16	1/16	1/16	1/16
X=4	1/16	1/16	1/16	1/16

Example: Weird Dice

$X=2 \quad Y=4$

Roll 2 fair 4-sided dice independently

Let U be the min value $U = \min(X, Y)$
Let W be the max value $W = \max(X, Y)$

$\Omega_U = \Omega_W = \{1,2,3,4\}$

The **joint support** is:

$\Omega_{U,W} = \{$ $(2,3)$ $(3,2)$

$p_{U,W}$	W=1	W=2	W=3	W=4
U=1				
U=2				
U=3				
U=4				

Example: Weird Dice

Roll 2 fair 4-sided dice independently

Let U be the min value $U = \min(X, Y)$
Let W be the max value $W = \max(X, Y)$

$\Omega_U = \Omega_W = \{1,2,3,4\}$

The **joint support** is:
$\Omega_{U,W} = \{(u, w) \in \Omega_U \times \Omega_W; u \leq w\}$

$p_{U,V}$	$W=1$	$W=2$	$W=3$	$W=4$
$U=1$				
$U=2$				
$U=3$				
$U=4$				

Example: Weird Dice

(1, 2)
2, 1

Roll 2 fair 4-sided dice independently

Let U be the min value $U = \min(X, Y)$
Let W be the max value $W = \max(X, Y)$

$\Omega_U = \Omega_W = \{1,2,3,4\}$

The **joint support** is:
$\Omega_{U,W} = \{(u, w) \in \Omega_U \times \Omega_W ; u \leq w\}$

What is the **joint PMF**?

$p_{U,W}(u, w) =$

$p_{U,V}$	W=1	W=2	W=3	W=4
U=1	1/16	2/16	2/16	2/16
U=2	0	1/16	2/16	
U=3	0	0	1/16	
U=4	0	0	0	1/16

Example: Weird Dice

Roll 2 fair 4-sided dice independently

Let U be the min value $U = \min(X, Y)$
Let W be the max value $W = \max(X, Y)$

$\Omega_U = \Omega_W = \{1,2,3,4\}$

The **joint support** is:
$\Omega_{U,W} = \{(u, w) \in \Omega_U \times \Omega_W; u \leq w\}$

$p_{U,V}$	W=1	W=2	W=3	W=4
U=1	1/16	2/16	2/16	2/16
U=2	0	1/16	2/16	2/16
U=3	0	0	1/16	2/16
U=4	0	0	0	1/16

What is the **joint PMF**?

$$p_{U,W}(u, w) = \begin{cases} 1/16 & \text{if } (u, w) \in \Omega_{U,W} \text{ and } u = w \\ 2/16 & \text{if } (u, w) \in \Omega_{U,W} \text{ and } u < w \\ 0 & \text{otherwise} \end{cases}$$

Example: Weird Dice

Roll 2 fair 4-sided dice independently

Let U be the min value $U = \min(X, Y)$
Let W be the max value $W = \max(X, Y)$

$\Omega_U = \Omega_W = \{1,2,3,4\}$

The **joint support** is:
$\Omega_{U,W} = \{(u, w) \in \Omega_U \times \Omega_W; u \leq w\}$

What is the **joint PMF?**

$$p_{U,W}(u, w) = \begin{cases} 1/16 & \text{if } (u, w) \in \Omega_{U,W} \text{ and } u = w \\ 2/16 & \text{if } (u, w) \in \Omega_{U,W} \text{ and } u < w \\ 0 & \text{otherwise} \end{cases}$$

$p_{U,V}$	W=1	W=2	W=3	W=4
U=1	1/16	2/16	2/16	2/16
U=2	0	1/16	2/16	2/16
U=3	0	0	1/16	2/16
U=4	0	0	0	1/16

U=2

U and W are **NOT independent** because $\Omega_{U,W} \neq \Omega_U \times \Omega_W$

Example: Weird Dice

Roll 2 fair 4-sided dice independently

Let U be the min value $U = \min(X, Y)$. Let W be the max value $W = \max(X, Y)$

Suppose I didn't know how to compute $p_U(u) = \mathbb{P}(U = u)$ *directly.*
Can we find it from the joint PMF $p_{U,W}(u, w)$*?*

We know $\mathbb{P}(U = 1 \cap W = w)$ for all w…

$\mathbb{P}(U = 1)$
$=$

$p_{U,V}$	W=1	W=2	W=3	W=4
U=1	1/16	2/16	2/16	2/16
U=2	0	1/16	2/16	2/16
U=3	0	0	1/16	2/16
U=4	0	0	0	1/16

Example: Weird Dice

Roll 2 fair 4-sided dice independently

Let U be the min value $U = \min(X, Y)$. Let W be the max value $W = \max(X, Y)$

Suppose I didn't know how to compute $p_U(u) = \mathbb{P}(U = u)$ *directly.*
Can we find it from the joint PMF $p_{U,W}(u, w)$*?*

We know $\mathbb{P}(U = 1 \cap W = w)$ for all w…

$$\mathbb{P}(U = 1)$$
$$= \mathbb{P}(U = 1 \cap W = 1) + \mathbb{P}(U = 1 \cap W = 2) + \mathbb{P}(U = 1 \cap W = 3) + \mathbb{P}(U = 1 \cap W = 4)$$
$$= \frac{1}{16} + \frac{2}{16} + \frac{2}{16} + \frac{2}{16} = \frac{7}{16}$$

Use LoTP because the events $V = 1$, $V = 2$, $V = 3$, $V = 4$ *partition the sample space*

$p_{U,V}$	W=1	W=2	W=3	W=4
U=1	1/16	2/16	2/16	2/16
U=2	0	1/16	2/16	2/16
U=3	0	0	1/16	2/16
U=4	0	0	0	1/16

Example: Weird Dice

Roll 2 fair 4-sided dice independently

Let U be the min value $U = \min(X, Y)$. Let W be the max value $W = \max(X, Y)$

Suppose I didn't know how to compute $p_U(u) = \mathbb{P}(U = u)$ directly.
Can we find it from the joint PMF $p_{U,W}(u, w)$?

In general…

$\mathbb{P}(U = u)$
$= \mathbb{P}(U = u \cap W = 1) + \mathbb{P}(U = u \cap W = 2)$
$\quad + \mathbb{P}(U = u \cap W = 3) + \mathbb{P}(U = u \cap W = 4)$
$= \sum_{w \in \Omega_W} p_{U,W}(u, w)$

Use LoTP because the events $W = 1$, $W = 2$, $W = 3$, $W = 4$ partition the sample space

$p_{U,V}$	W=1	W=2	W=3	W=4
U=1	1/16	2/16	2/16	2/16
U=2	0	1/16	2/16	2/16
U=3	0	0	1/16	2/16
U=4	0	0	0	1/16

Example: Weird Dice

Roll 2 fair 4-sided dice independently

Let U be the min value $U = \min(X, Y)$. Let W be the max value $W = \max(X, Y)$

Suppose I didn't know how to compute $p_U(u) = \mathbb{P}(U = u)$ directly.
Can we find it from the joint PMF $p_{U,W}(u, w)$?

In general…

$$\mathbb{P}(U = u) = \sum_{w \in \Omega_W} p_{U,W}(u, w)$$

Use LoTP because the events $W = 1$, $W = 2$, $W = 3$, $W = 4$ ***partition the sample space***

$p_U(u)$ is the "marginal" PMF for U
(because we "marginalized" W)

$p_{U,V}$	W=1	W=2	W=3	W=4
U=1	1/16	2/16	2/16	2/16
U=2	0	1/16	2/16	2/16
U=3	0	0	1/16	2/16
U=4	0	0	0	1/16

Marginal Distribution

If you have the joint PMF for two random variables, you can find the PMF of one of them by **using the law of total probability**, **partitioning on the values of the other random variable:**

$$p_X(a) = \sum_{b \in \Omega_Y} p_{X,Y}(a, b)$$

^ it's the same PMF we've talked about before, the "marginal" is just there to indicate is was derived from a joint PMF

Example: Weird Dice

Roll 2 fair 4-sided dice independently

Let U be the min value $U = \min(X, Y)$. Let W be the max value $W = \max(X, Y)$

Suppose I didn't know how to compute $p_U(u) = \mathbb{P}(U = u)$ *directly.*
Can we find it from the joint PMF $p_{U,W}(u, w)$*?*

$$p_U(u) = \begin{cases} \frac{7}{16} & \text{if } u = 1 \\ \frac{5}{16} & \text{if } u = 2 \\ \frac{3}{16} & \text{if } u = 3 \\ \frac{1}{16} & \text{if } u = 4 \\ 0 & \text{otherwise} \end{cases}$$

$p_{U,V}$	W=1	W=2	W=3	W=4
U=1	1/16	2/16	2/16	2/16
U=2	0	1/16	2/16	2/16
U=3	0	0	1/16	2/16
U=4	0	0	0	1/16

Joint **Expectation**

Joint Expectations

For a function $g(X, Y)$, the expectation can be written in terms of the joint PMF.

$$\mathbb{E}[g(X,Y)] = \sum_{(a,b)\in\Omega_{X,Y}} g(a,b) \cdot p_{X,Y}(a,b)$$

Same ideas as before!

Examples of joint functions:

$g(X,Y) = X + Y$, $g(X,Y) = XY$, $g(X,Y) = X^Y$, $g(X,Y) = \binom{X}{Y}$, *etc.*

Joint Distributions

For **continuous** joint distributions, everything will look very similar but we *replace summations with integrals*, and use a *density function instead of the PMF*

We have two **continuous** random variables X and Y
(that may or may not be independent)

Joint **Support/Range** - $\Omega_{X,Y}$

$\mathbb{R} \times \mathbb{R}$

$\Omega_{X,Y} = \{(a, b) : f_{X,Y}(a, b) > 0\} \subseteq \Omega_X \times \Omega_Y$

Joint **PDF** - $f_{X,Y}(a, b)$

$f_{X,Y}(a, b)$ *defined for* ***all*** $(a, b) \in \mathbb{R} \times \mathbb{R}$

Joint **CDF** - $F_{X,Y}(a, b)$

$F_{X,Y}(a, b) = \mathbb{P}(X \leq a, Y \leq b)$
defined for ***all*** $(a, b) \in \mathbb{R} \times \mathbb{R}$

Normalization Property:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x, y) dx\, dy = 1$$

Joint **Independence**

> $f_{X,Y}(a, b) = f_X(a) \cdot f_Y(b)$ for all $(a, b) \in \Omega_{X,Y}$
> $\Omega_{X,Y} = \Omega_X \times \Omega_Y$

Joint **Expectation**

$\mathbb{E}[g(X, Y)] =$
$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y)\, f_{X,Y}(x, y) dx\, dy$

Marginal **PDF** - $f_X(x)$, $f_Y(y)$

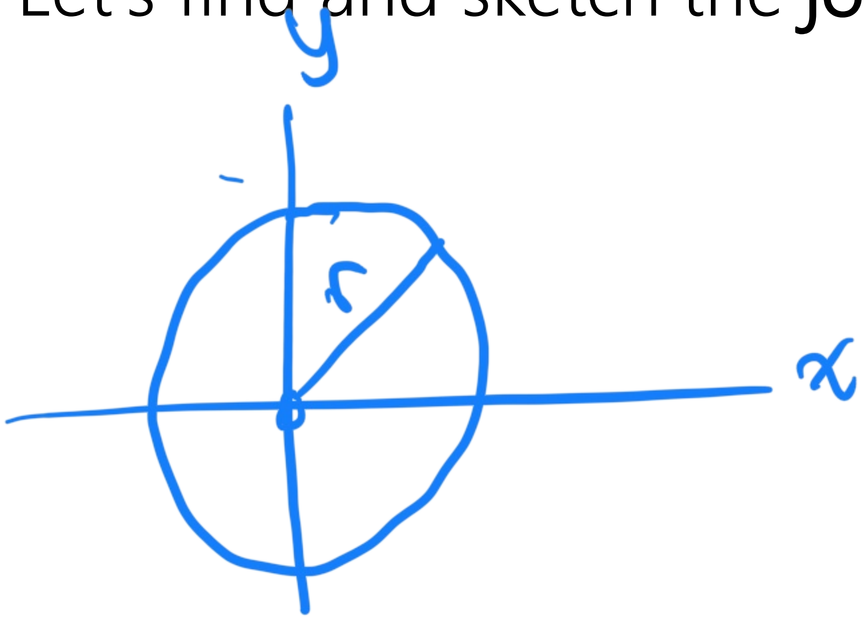
$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dy$
$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dx$

Notice we're integrating (summing) over what the other RV can be

Example: Darts

We throw a dart uniformly at random onto a circle of radius R centered around the version. X and Y are the x and y coordinates of the point the dart lands at.

Let's find and sketch the **joint range** $\Omega_{X,Y}$



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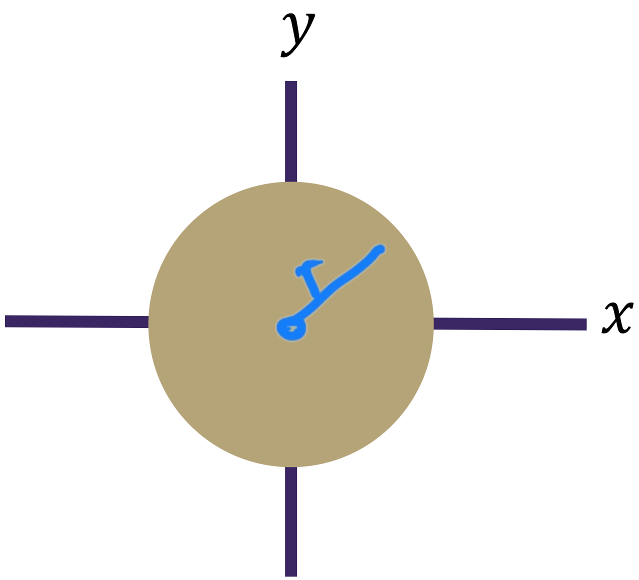
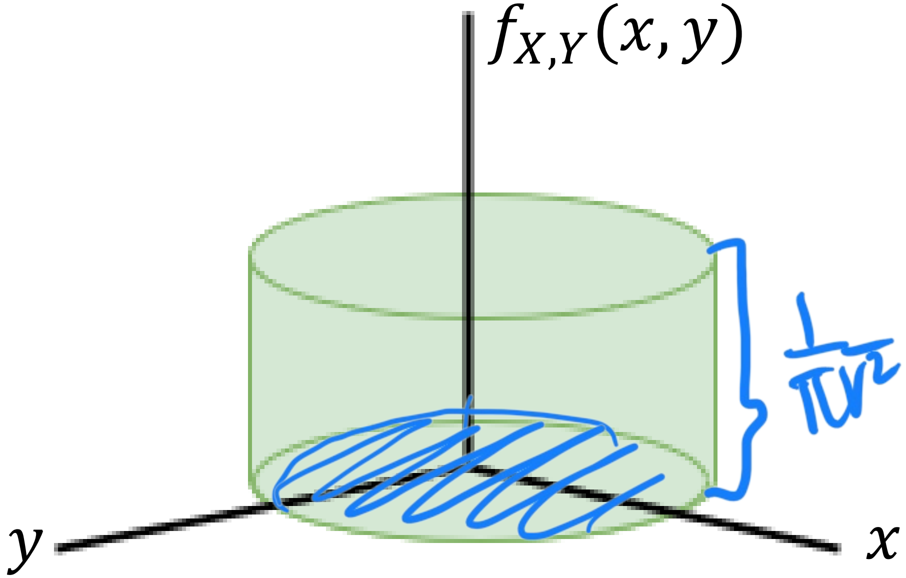


$$\Omega_{X,Y} = \{(x,y) \in \mathbb{R} \times \mathbb{R};\, \mathrm{x}^2 + \mathrm{y}^2 \leq r\}$$

Example: Darts

We throw a dart uniformly at random onto a circle of radius r centered around the version. X and Y are the x and y coordinates of the point the dart lands at.

Let's find and sketch the **joint PDF** $f_{X,Y}(x, y)$

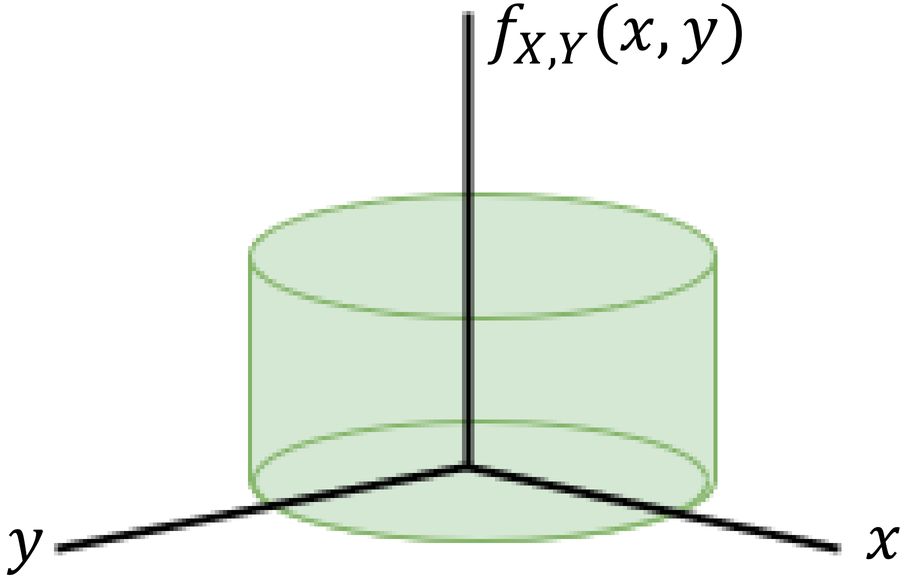


$$f_{X,Y}(x, y) = \begin{cases} \frac{1}{\pi r^2} & x^2 + y^2 \le r \text{ , on the circle} \\ 0 & \text{otherwise} \end{cases}$$

Example: Darts

We throw a dart uniformly at random onto a circle of radius r centered around the version. X and Y are the x and y coordinates of the point the dart lands at.

Let's find and sketch the **joint PDF** $f_{X,Y}(x,y)$



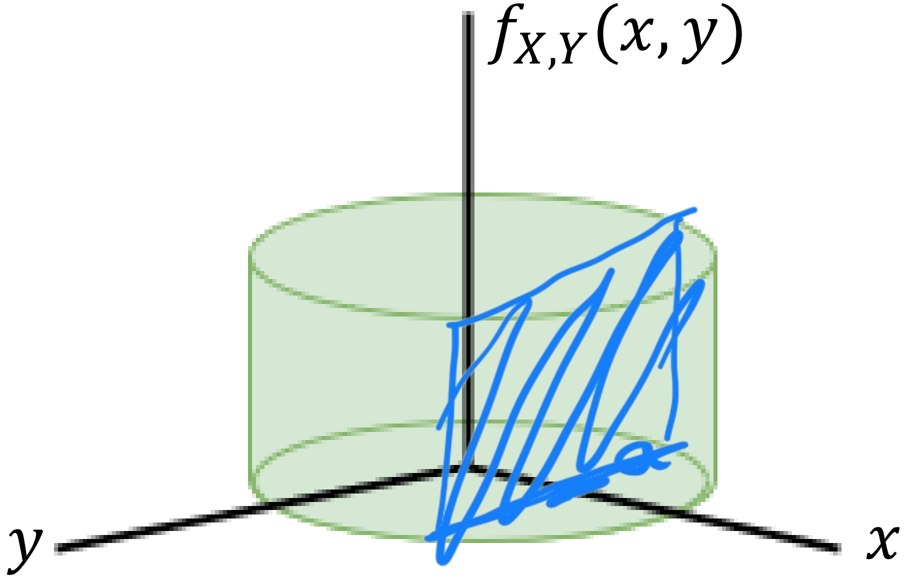
$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{\pi r^2} & x^2 + y^2 \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Example: Darts

We throw a dart uniformly at random onto a circle of radius r centered around the version. X and Y are the x and y coordinates of the point the dart lands at.

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{\pi r^2} & x^2 + y^2 \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

What is the marginal PDF for X and Y?



$f_X(a) =$

$f_Y(b) =$

Example: Darts

We throw a dart uniformly at random onto a circle of radius r centered around the version. X and Y are the x and y coordinates of the point the dart lands at.

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{\pi r^2} & x^2 + y^2 \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

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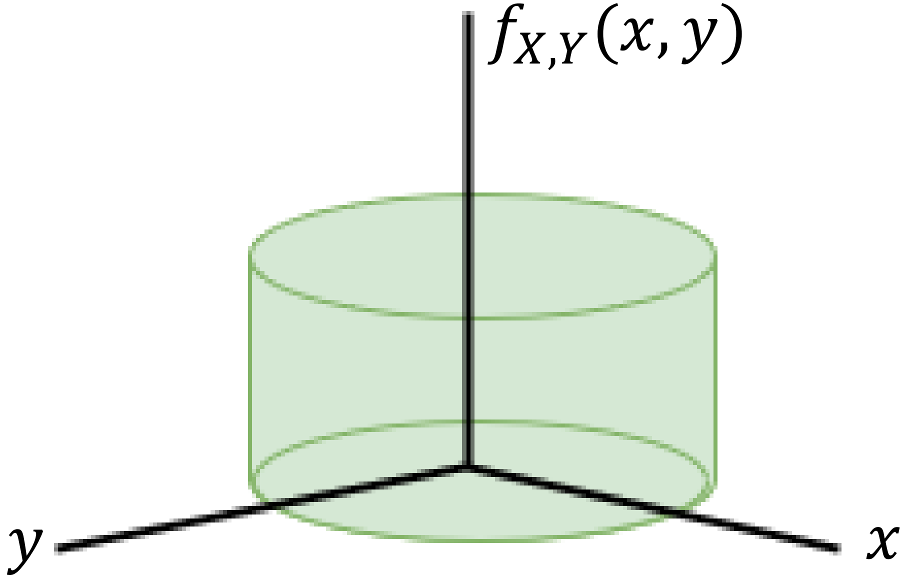


$$f_X(a) = \int_{-\infty}^{\infty} f_{X,Y}(a,y)\,dy$$

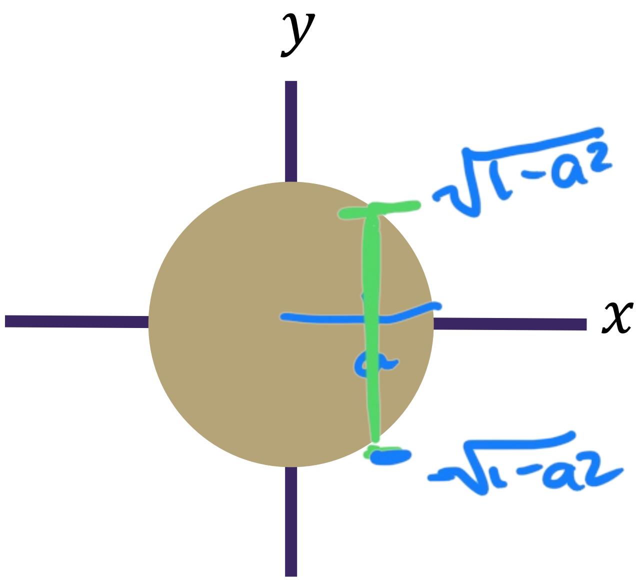
$$f_Y(b) = \int_{-\infty}^{\infty} f_{X,Y}(x,b)\,dx$$

Example: Darts

We throw a dart uniformly at random onto a circle of radius r centered around the version. X and Y are the x and y coordinates of the point the dart lands at.

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{\pi r^2} & x^2 + y^2 \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

What is the marginal PDF for X and Y?



$$f_X(a) = \int_{-\infty}^{\infty} f_{X,Y}(a,y)dy$$

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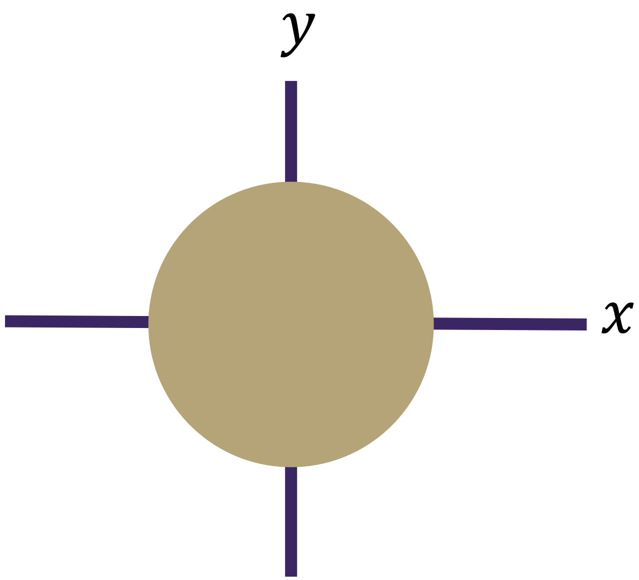
What is the marginal PDF for X and Y?



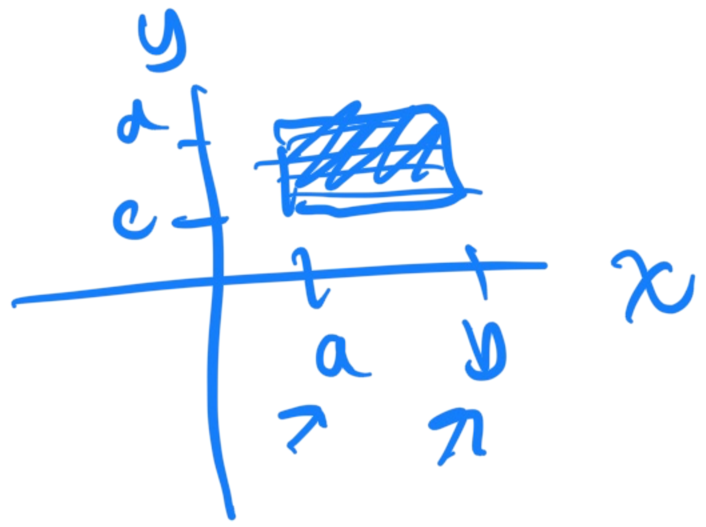
$$f_X(a) = \int_{-\infty}^{\infty} f_{X,Y}(a,y)dy = \int_{-\sqrt{r-a^2}}^{\sqrt{r-a^2}} \frac{1}{\pi r^2} dy = \frac{2\sqrt{r-a^2}}{\pi r^2}$$

$$f_Y(b) = \int_{-\infty}^{\infty} f_{X,Y}(x,b)dx = \int_{-\sqrt{r-b^2}}^{\sqrt{r-b^2}} \frac{1}{\pi r^2} dx = \frac{2\sqrt{r-b^2}}{\pi r^2}$$

Joint Continuous Probabilities

Just like we've done with PDFs for single random variables…

$$\mathbb{P}(a \leq X \leq b \cap c \leq Y \leq d) = \int_a^b \int_c^d f_{X,Y}(x,y)\,dy\,dx$$

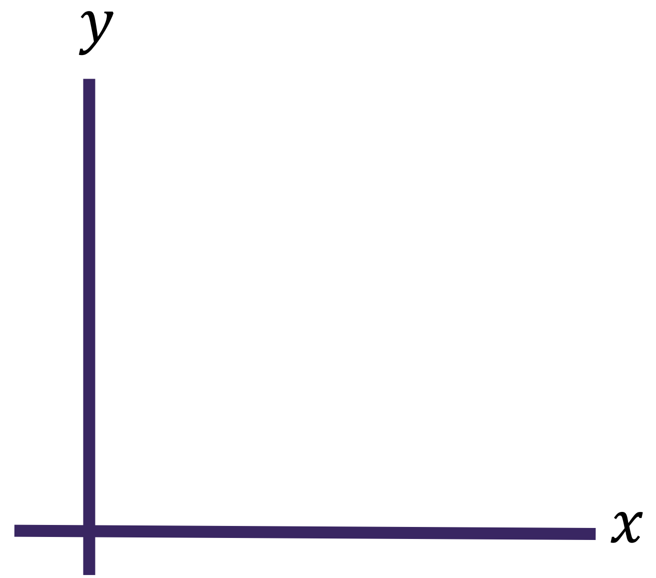


Example: Continuous Servers

The time until server 1 crashes is $X \sim \mathrm{Exp}(u)$, and the time until server 2 crashes is $Y \sim \mathrm{Exp}(v)$. Both servers are independent of each other.

What is the probability server 1 crashes before server 2?

$\mathbb{P}(X < Y) =$



Example: Continuous Servers

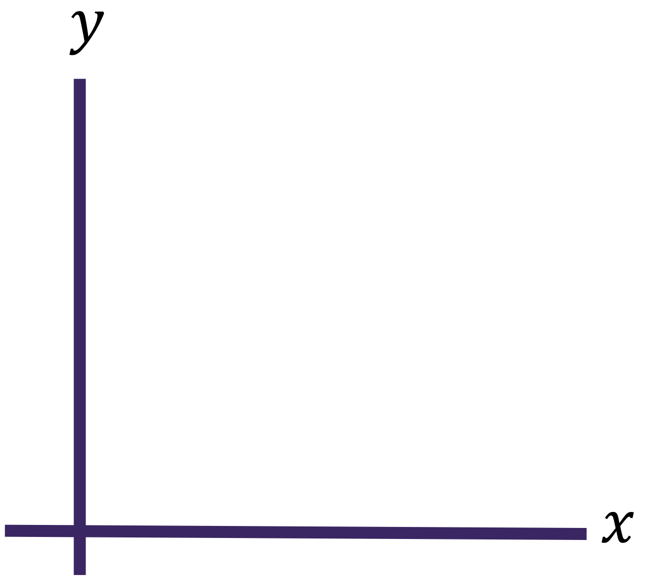
The time until server 1 crashes is $X \sim \mathrm{Exp}(u)$, and the time until server 2 crashes is $Y \sim \mathrm{Exp}(v)$. Both servers are independent of each other.

What is the probability server 1 crashes before server 2?

$\mathbb{P}(X < Y)$

$= \int_0^\infty \int_x^\infty f_{X,Y}(x,y)\, dy\, dx$

$= \int_0^\infty \int_x^\infty f_X(x) f_Y(y)\, dy\, dx$ *by independence*



Analogues for continuous

Everything we saw today has a continuous version.

There are "no surprises"– replace pmf with pdf and sums with integrals.

	Discrete	Continuous
Joint PMF/PDF	$p_{X,Y}(x,y) = P(X=x, Y=y)$	$f_{X,Y}(x,y) \neq P(X=x, Y=y)$
Joint CDF	$F_{X,Y}(x,y) = \sum_{t \le x} \sum_{s \le y} p_{X,Y}(t,s)$	$F_{X,Y}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(t,s) ds dt$
Normalization	$\sum_x \sum_y p_{X,Y}(x,y) = 1$	$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y) dx dy = 1$
Marginal PMF/PDF	$p_X(x) = \sum_y p_{X,Y}(x,y)$	$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) dy$
Expectation	$E[g(X,Y)] = \sum_x \sum_y g(x,y) p_{X,Y}(x,y)$	$E[g(X,Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y) dx dy$
Independence	$\forall x, y, p_{X,Y}(x,y) = p_X(x) p_Y(y)$	$\forall x, y, f_{X,Y}(x,y) = f_X(x) f_Y(y)$

Covariance

We sometimes want to measure how "intertwined" X and Y are – how much knowing about one of them will affect the other.

If X turns out "big" how likely is it that Y will be "big" how much do they "vary together"

Covariance

$$\mathbf{Cov}(\mathbf{X}, \mathbf{Y}) = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])] = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$$

Covariance

$\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y) + 2\text{Cov}(X, Y)$

That's consistent with our previous knowledge for independent variables. (for X, Y independent, $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$).

You and your friend are playing a game, you flip a coin: if heads you pay your friend a dollar, if tails they pay you a dollar. Let X be your profit and Y be your friend's profit.

What is $\text{Var}(X + Y)$?

Before you calculate, make a prediction. What should it be?

Covariance

You and your friend are playing a game, you flip a coin: if heads you pay your friend a dollar, if tails they pay you a dollar. Let X be your profit and Y be your friend's profit.

What is $\text{Var}(X + Y)$?

$\text{Var}(X) = \text{Var}(Y) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = 1 - 0^2 = 1$

$\text{Cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$

$\mathbb{E}[XY] = \frac{1}{2} \cdot (-1 \cdot 1) + \frac{1}{2}(1 \cdot -1) = -1$

$\text{Cov}(X, Y) = -1 - 0 \cdot 0 = -1.$

$\text{Var}(X + Y) = 1 + 1 + 2 \cdot -1 = 0$