# Discrete Random Variable Zoo II

CSE 312 24Su

Lecture 11

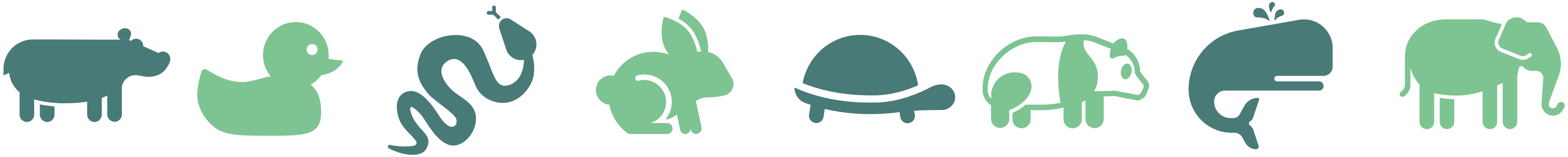
# Logistics

- Coding part for HW3 due tonight
- Couple extra office hours this weekend before the midterm
- Review session today at 4pm
- Review session during lecture slot on Monday
  - Bring questions/topics you want to review!!!

# Discrete Zoo of Random Variables

There are *common patterns* of random experiments

We're going to **identify some common patterns**, and **compute the support, PMF, CDF, expectation, and variance** for them, so *when we see a random variable that matches that pattern*, we don't have to re-compute everything!



# Discrete ***Uniform*** Distribution

$X \sim \mathrm{Unif}(a, b)$

$X$ is a uniformly random integer between $a$ and $b$ (inclusive)

Parameter $a$ is the minimum value in the support, $b$ is the maximum value in the support.

PMF: $p_X(k) = \frac{1}{b-a+1}$ for $k \in \mathbb{Z}$, $a \leq k \leq b$

CDF: $F_X(k) = \frac{k-a+1}{b-a+1}$ for $k \in \mathbb{Z}$, $a \leq k \leq b$.

Expectation: $\mathbb{E}[X] = \frac{a+b}{2}$

Variance: $\mathrm{Var}(X) = \frac{(b-a)(b-a+2)}{12}$

# ***Bernoulli*** Distribution

$X \sim \text{Ber}(p)$

$X$ is the indicator random variable that the trial was a success.

Parameter $p$ is probability of success on the trial.

**PMF**: $p_X(0) = 1 - p, p_X(1) = p$

**CDF**: $F_X(k) = \begin{cases} 0 & \text{if } k < 0 \\ 1 - p & \text{if } 0 \leq k < 1 \\ 1 & \text{if } k \geq 1 \end{cases}$

**Expectation**: $\mathbb{E}[X] = p$

**Variance**: $\text{Var}(X) = p(1 - p)$

# ***Binomial*** Distribution

$X \sim \text{Bin}(n, p)$

$X$ is the number of successes across $n$ <u>independent</u> trials.

$n$ is the number of independent trials.
$p$ is the probability of success for one trial.

PMF: $p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}$ for $k \in \{0,1,\ldots,n\}$

CDF: $F_X$ is ugly.

Expectation: $\mathbb{E}[X] = np$

Variance: $\text{Var}(X) = np(1-p)$

# Discrete Zoo of Random Variables

- **Uniform:** Every integer between $a$ and $b$ are equally likely
  $\text{Un}if(a, b)$
- **Bernoulli:** Whether there is success in one trial
  $\text{Ber}(p)$ is 1 with probability $\boldsymbol{p}$ and 0 otherwise
- **Binomial:** Number of successes in $n$ independent trials
  $\text{Bin}(n, p)$ - $n$ independent trials, probability $p$ of success on each trial

# **Example:** Unpopular Donuts

A donut shop serves 50 people a day and serves a *mango chili lime* donut. The probability that a customer chooses this donut is 0.2. All customers' choices are independent of each other.

*What is the probability that exactly* $10$ *people choose this flavor?*

*What is the probability that at least* $3$ *people choose this flavor?*

# **Example:** Unpopular Donuts

A donut shop serves 50 people a day and serves a *mango chili lime* donut. The probability that a customer chooses this donut is 0.2. All customers' choices are independent of each other.

*What is the probability that exactly* $10$ *people choose this flavor?*

PMF: $p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}$ for $k \in \{0, 1, \ldots, n\}$

*What is the probability that at least* $3$ *people choose this flavor?*

# **Example:** Unpopular Donuts

A donut shop serves 50 people a day and serves a *mango chili lime* donut. The probability that a customer chooses this donut is 0.2. All customers' choices are independent of each other.

*What is the probability that* <u>*exactly*</u> $10$ *people choose this flavor?*

$X\sim$ number of people who choose this flavor. $X \sim \mathrm{Bin}(50, 0.2)$

$\mathbb{P}(X = 10) = \binom{50}{10} 0.2^{10} (1 - 0.2)^{50-10}$

*What is the probability that* <u>*at least*</u> $3$ *people choose this flavor?*

$$\mathbb{P}(X \geq 3) = 1 - \mathbb{P}(X < 3) = 1 - (\mathbb{P}(X = 0) + \mathbb{P}(X = 1) + \mathbb{P}(X = 2))$$
$$= 1 - (\binom{50}{0} 0.2^0 (0.8)^{50-0} + \binom{50}{1} 0.2^1 (0.8)^{50-1} + \binom{50}{2} 0.2^2 (0.8)^{50-2})$$

# Discrete Zoo of Random Variables (today!)

- **Uniform:** Every integer between $a$ and $b$ are equally likely
  $\mathrm{Unif}(a,b)$
- **Bernoulli:** Whether there is success in one trial
  $\mathrm{Ber}(p)$ is 1 with probability $p$ and 0 otherwise
- **Binomial:** Number of successes in $n$ independent trials
  $\mathrm{Bin}(n,p)$ - $n$ independent trials, probability $p$ of success on each trial
- **Geometric:** Number of trials till first success
  $\mathrm{Geo}(p)$ - probability $p$ of success on each trial
- **Poisson:** Number of successes in a time interval
  $\mathrm{Poi}(\lambda)$ - average number of successes in the time interval
- **Negative Binomial:** Number of trials till $r$'th success
  $\mathrm{NegBin}(r,p)$ - probability $p$ of success on each trial, want trials till the $r$'th success
- **Hypergeometric:** Number of successes when drawing a sample
  $\mathrm{HypGeo}(N,K,n)$ - drawing a sample of $n$ items from a set of $N$ with $K$ successes

# Situation: Geometric

How many *independent* trials are needed until the first success?

*Familiar Example:*

You flip a coin (which comes up heads with probability $p$) independently until you get a heads. How many flips did you need?

# Geometric Distribution

$X \sim \text{Geo}(p)$

$X$ is the number of trials needed to see the first success.

$p$ is the probability of success for one trial.

# Geometric Distribution *Examples*

How many bits can we write *before one is incorrect*?

How many questions do you have to answer *until you get one right*?

How many times can you run an experiment *until it fails for the first time*?

# Geometric Distribution

$X \sim \text{Geo}(p)$

$X$ is the number of trials needed to see the first success.

$p$ is the probability of success for one trial.

$\Omega_X = \{1,2,3,4, \ldots\}$

PMF: $p_X(k) = (1-p)^{k-1}p$ for $k \in \{1,2,3, \ldots\}$

CDF: $F_X(k) = 1-(1-p)^k$ for $k \in \mathbb{N}$

Expectation: $\mathbb{E}[X] = \frac{1}{\mathrm{p}}$

Variance: $\text{Var}(X) = \frac{1-p}{p^2}$

# Geometric: Analysis

Both the expectation and variance are new to us.

The derivations of both are uninformative

Every derivation I've ever seen has wild algebra tricks.

# Geometric: Expectation

$\mathbb{E}[X] = \sum_{k=1}^{\infty} k(1-p)^{k-1}p$

$= p\sum_{k=1}^{\infty} k(1-p)^{k-1} = p \cdot \frac{1}{p^2} = \frac{1}{p}.$

Intuition: Smaller $p$ means longer wait

$\text{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$

$= \frac{2-p}{p^2} - \frac{1}{p^2} = \frac{1-p}{p^2}$

Intuition: for small $p$ lots of variance (might have to wait a long time, and it's variable)

For large $p$ very little variance (for $p = 1$ there's no variation at all!)

# Geometric Property

Suppose you're flipping coins independently until you see a heads.

$X \sim \mathbf{Geo}(p)$ is number of flips till the first head

> The first three came up tails.

> $Y$ is number of flips *left* until you see the first head

*Does $Y$ also follow $\mathbf{Geo}(p)$?*

T T T ? ? ? ? …

Fill out the poll everywhere:
pollev.com/cse312

# Geometric Property - *Memoryless*

Geometric random variables are called "**memoryless**"

Suppose you're flipping coins independently until you see a heads.

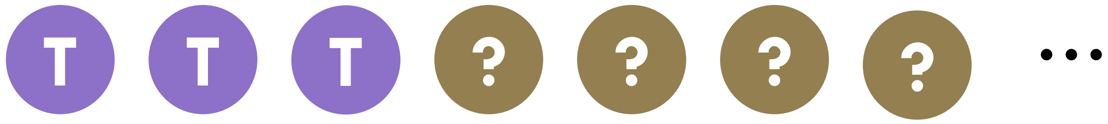
$X \sim \text{Geo}(p)$ is number of flips till the first head

> The first three came up tails.

> $Y$ is number of flips *left* until you see the first head *after the first 3 tails*

*Does $Y$ also follow $\text{Geo}(p)$?* Yes!

The coin "forgot" it already came up tails 3 times.



# Formally…

Let $X$ be the number of flips needed, $Y$ be the flips after the third.

$\mathbb{P}(Y = k | X \geq 3) = \mathbb{P}(Y = k \cap X \geq 3)/\mathbb{P}(X \geq 3)$

$\frac{(1-p)^{k+3-1}p}{(1-p)^3}$

$= (1-p)^{k-1}p$

Which is $p_X(k)$.

# Formally…

Let $X$ be the number of flips needed, $Y$ be the flips after the third.

$\mathbb{P}(Y = k | X \geq 3) = \mathbb{P}(Y = k \cap X \geq 3)/\mathbb{P}(X \geq 3)$

$\frac{(1-p)^{k+3-1}p}{(1-p)^3}$

$= (1-p)^{k-1}p$

Which is $p_X(k)$.

A geometric distribution is **memoryless:**

> If $X \sim \text{Geo}(p)$
> $\mathbb{P}(X \geq a + b | X \geq a) = \mathbb{P}(X \geq b)$

# Scenario: The Poisson Distribution

We're trying to count the **number of times something happens in some interval of <u>time</u>**.

> We know the average number that happen (i.e. the expectation)

> Each occurrence is independent of the others.

> There are a VERY large number of "potential sources" for those events, few of which happen.

# Scenario: The Poisson Distribution

We're trying to count the **number of times something happens in some interval of time**.

*Example of situation that's hard to model without a Poisson distribution*

We want to model **number of people who buy a mango chili lime donut in a day**

> We did this with a binomial distribution, and said there are 50 people, each who have probability 0.2 of buying the donut -> $\text{Bin}(50,0.2)$

> Realistically though, there are **way** more people who could possibly come into the donut shop, and it's very hard to model the probability of each person choosing to come into the shop and buy the donut today

> With a **Poisson distribution** we can model this when all we know is the *average number of people buying that donut in a day from historical data*

# The Poisson Distribution

Classic applications:

How many traffic accidents occur in Seattle in a day

How many major earthquakes occur in a year (not including aftershocks)

How many customers visit a bakery in an hour

Why not just use counting coin flips?

What are the flips…the number of cars? Every person who might visit the bakery?
**There are way too many of these to count exactly or think about dependency between. But a Poisson might accurately model what's happening.**

# It’s a model – it’s doesn’t *fully* reflect the real world

By modeling choice, we mean that we’re choosing math that we think represents the real world as best as possible

Is every traffic accident really independent?

Not *really*, one causes congestion, which causes angrier drivers. Or both might be caused by bad weather/more cars on the road.

But we assume they are (because the dependence is so weak that the model is useful).

# Poisson Distribution

$X \sim \text{Poi}(\lambda)$

$X$ is the number of incidents seen in a particular time interval.

Let $\lambda$ be the average number of incidents in that time interval.

Support: $\mathbb{N}$ (all *natural* numbers)

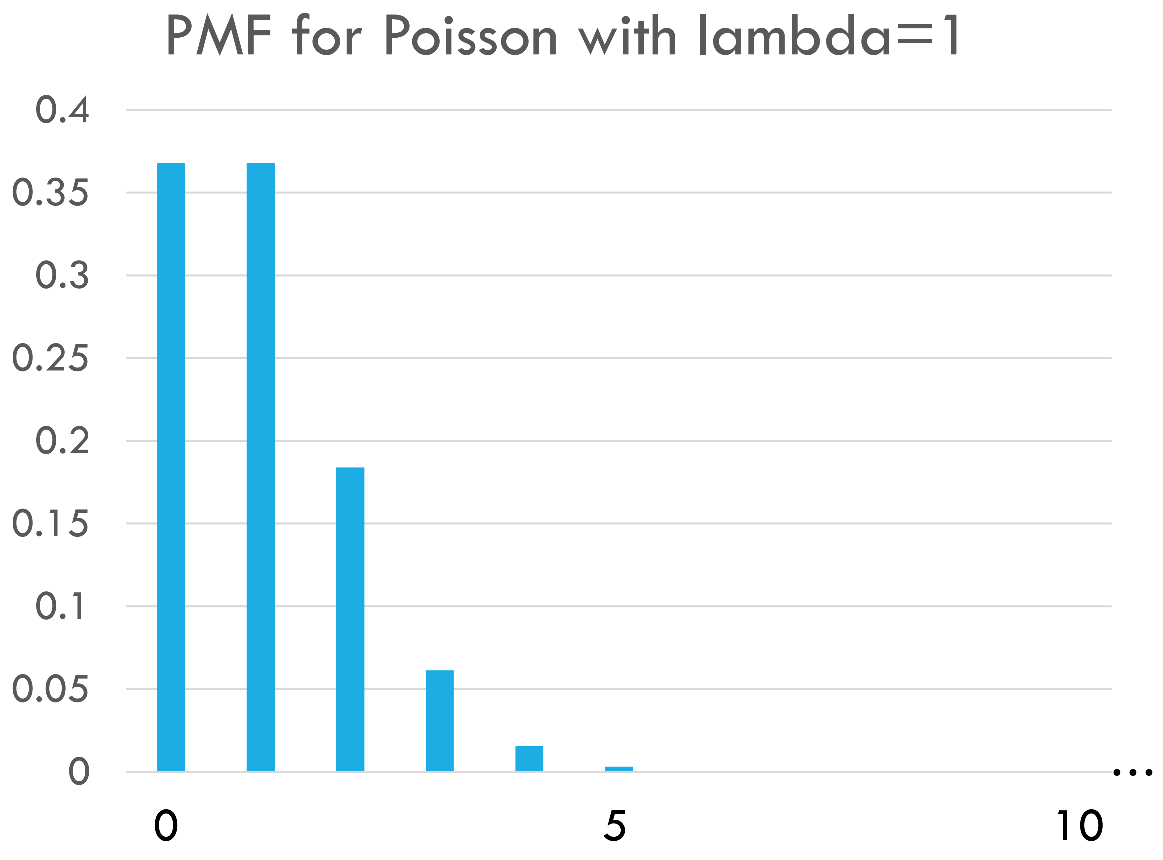
PMF: $p_X(k) = \frac{\lambda^k e^{-\lambda}}{k!}$ (for $k \in \mathbb{N}$)

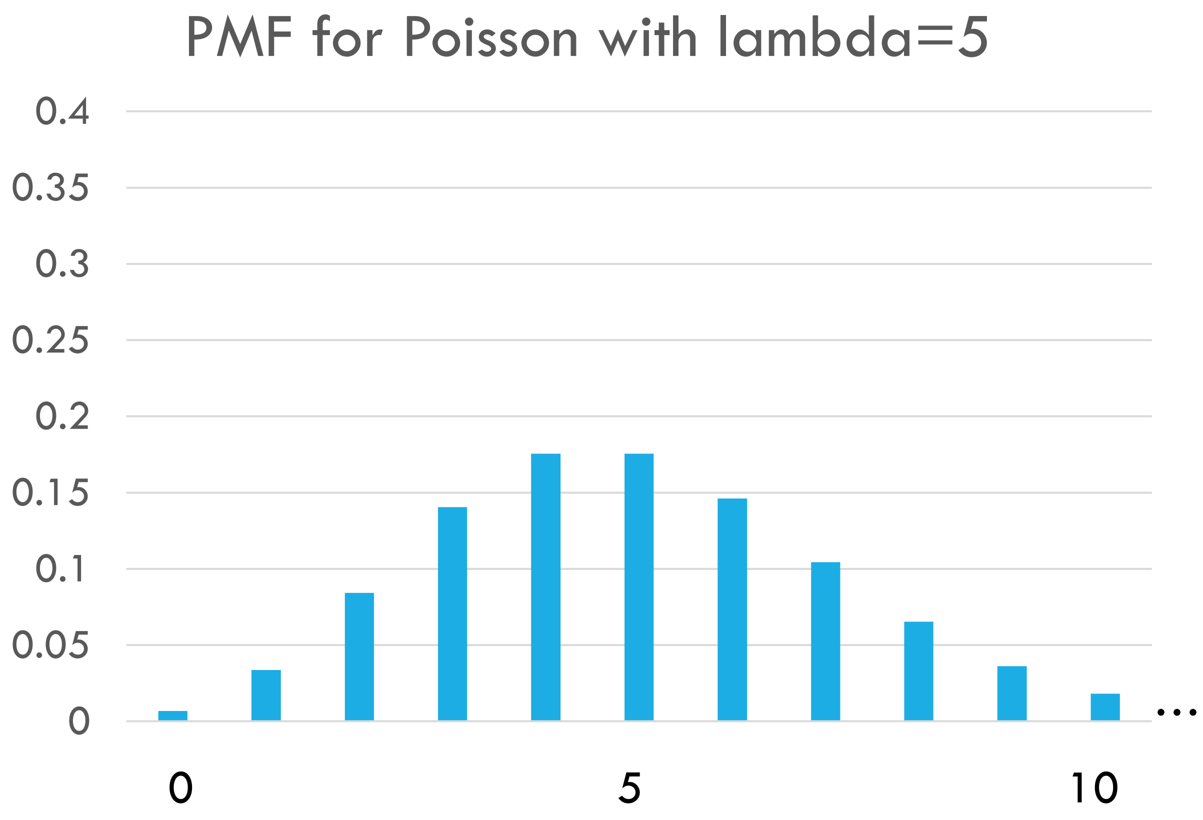
CDF: $F_X(k) = e^{-\lambda} \sum_{i=0}^{\lfloor k \rfloor} \frac{\lambda^i}{i!}$

Expectation: $\mathbb{E}[X] = \lambda$

Variance: $\text{Var}(X) = \lambda$

# Poisson Distribution (sample PMFs)



# Let's take a closer look at that PMF

$p_X(k) = \frac{\lambda^k e^{-\lambda}}{k!}$ (for $k \in \mathbb{N}$)

If this is a real PMF, it should sum to 1.

$\sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!}$

$= e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}$

*Taylor Series for $e^x$*

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x$$

$= e^{-\lambda} e^{\lambda} = e^0 = 1$

# Let's check something…the expectation

$\mathbb{E}[X] = \sum_{k=0}^{\infty} k \cdot e^{-\lambda} \frac{\lambda^k}{k!}$

$= \sum_{k=1}^{\infty} k \cdot e^{-\lambda} \frac{\lambda^k}{k!}$ first term is $0$.

$= \sum_{k=1}^{\infty} e^{-\lambda} \frac{\lambda^k}{(k-1)!}$ cancel the $k$.

$= \lambda \sum_{k=1}^{\infty} e^{-\lambda} \frac{\lambda^{k-1}}{(k-1)!}$ factor out $\lambda$.

$= \lambda \sum_{j=0}^{\infty} e^{-\lambda} \frac{\lambda^j}{(j)!}$ Define $j = k - 1$

$= \lambda \cdot 1$ The summation is just the pmf!

# Where did this expression come from?

**$X$ is the number of car accidents in a day**

If we knew the exact number of cars, and they all had identical probabilities of causing an accident…

It'd be just like counting the number of heads in $n$ flips of a bunch of coins (the coins are just VERY biased).

The Poisson is a certain limit as $n \to \infty$ but $np$ (the expected number of accidents) stays constant.

# Scenario: Negative Binomial

**Run <u>independent</u> trials with probability $p$. How many trials do you need *until $r$ successes*?**

*Example*

You're playing a carnival game, and there are $r$ little kids nearby who all want a stuffed animal. You can win a single game (and thus win one stuffed animal) with probability $p$ (independently each time) How many times will you need to play the game before every kid gets their toy?

# Try it

**Run <u>independent</u> trials with probability $p$. How many trials do you need *until $r$ successes*?**

$X$ is the number of trials till (and including) the r'th success

What is the **support** of $X$?

What's the **PMF**?

i.e., what is the probability it takes exactly $k$ trials till the r'th success?

Fill out the poll everywhere: pollev.com/cse312

# Try it

**Run independent trials with probability $\boldsymbol{p}$. How many trials do you need *until* $\boldsymbol{r}$ *successes*?**

$X$ is the number of trials till (and including) the r'th success

What is the **support** of $X$? $\Omega_X = \{r, r+1, r+2, \ldots\}$

What's the **PMF**?

i.e., what is the probability it takes exactly $k$ trials till the r'th success?

# Negative Binomial Analysis

Run independent trials with probability $p$
$X$ is the number of trials till (and including) the r'th success

What's the **PMF**? Well how would we know $X = k$?

# Negative Binomial Analysis

What's the **PMF**? Well how would we know $X = k$?

Of the first $k-1$ trials, $r-1$ must be successes.
And trial $k$ must be a success.

1. We want **exactly $\boldsymbol{r-1}$ of the first $\boldsymbol{k-1}$ to be successes** – this sounds like a binomial! It's the $p_Y(r-1)$ where $Y \sim \text{Bin}(k-1, r-1)$:
$\binom{k-1}{r-1}(1-p)^{k-1-(r-1)}p^{r-1} = \binom{k-1}{r-1}(1-p)^{k-r}p^{r-1}$

2. Multiply by $p$, probability **$\boldsymbol{k}$'th trial is success**

Total: $p_X(k) = \binom{k-1}{r-1}(1-p)^{k-r}p^r$

# Negative Binomial Analysis

*$X$ is the number of trials till we see $r$ successes*

To see $r$ successes:

We do trials until we see success $1$.

Then do trials until success $2$.

...do trials until success $r$.

What's the **expectation** and **variance** (hint: linearity)?

How can we write $X$ as a sum of random variables?

Fill out the poll everywhere: pollev.com/cse312

# Negative Binomial Analysis

*$X$ is the number of trials till we see $r$ successes*

To see $r$ successes:

We do trials until we see success $1$.

Then do trials until success $2$.

…do trials until success $r$.

The total number of flips is…the sum of geometric random variables!

# Negative Binomial Analysis

Let $Z_1, Z_2, \ldots, Z_r$ be independent copies of $\text{Geo}(p)$

$Z_i$ are called "independent and identically distributed" or "i.i.d.'

Because they are independent…and have identical pmfs.

$$X \sim \text{NegBin}(r, p)\ X = Z_1 + Z_2 + \cdots + Z_r.$$

$$\mathbb{E}[X] = \mathbb{E}[Z_1 + Z_2 + \cdots Z_r] = \mathbb{E}[Z_1] + \mathbb{E}[Z_2] + \cdots + \mathbb{E}[Z_r] = r \cdot \frac{1}{p}$$

# Negative Binomial Analysis

Let $Z_1, Z_2, \dots, Z_r$ be independent copies of $\mathtt{Geo}(p)$

$X \sim \text{NegBin}(r, p)$ $X = Z_1 + Z_2 + \cdots + Z_r$.

$\text{Var}(X) = \text{Var}(Z_1 + Z_2 + \cdots + Z_r)$

Up until now we've just used the observation that $X = Z_1 + \cdots + Z_r$.

$= \text{Var}(Z_1) + \text{Var}(Z_2) + \cdots + \text{Var}(Z_r)$ **because the $\boldsymbol{Z_i}$ are independent.**

$= r \cdot \frac{1-p}{p^2}$

# Negative Binomial

$X \sim \mathrm{NegBin}(\mathrm{r}, \mathrm{p})$

Parameters: $r$: the number of successes needed, $p$ the probability of success in a single trial

$X$ is the number of trials needed to get the $r^{\text{th}}$ success.

**PMF:** $p_X(k) = \binom{k-1}{r-1}(1-p)^{k-r}p^r$

**CDF:** $F_X(k)$ is ugly, don't bother with it.

**Expectation:** $\mathbb{E}[X] = \frac{r}{p}$

**Variance:** $\mathrm{Var}(\mathrm{X}) = \frac{\mathrm{r}(1-\mathrm{p})}{p^2}$

# Scenario: Hypergeometric

You have an urn with $N$ balls, of which $K$ are purple. You are going to draw $n$ balls out of the urn **without** replacement. How many purple balls do we get in this sample?

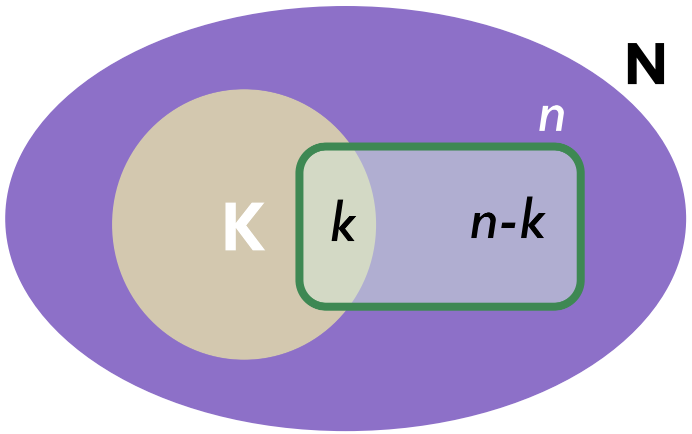
$X$ is the number of purple balls in this sample

# Hypergeometric: Analysis (PMF)

You have an urn with $N$ balls, of which $K$ are purple. You are going to draw $n$ balls out of the urn **without** replacement. How many purple balls do we get in this sample?

$X$ is the number of purple balls in this sample

**If you draw out $n$ balls, what is the probability you see $k$ purple ones?**



# Hypergeometric: Analysis (PMF)

You have an urn with $N$ balls, of which $K$ are purple. You are going to draw $n$ balls out of the urn **without** replacement. How many purple balls do we get in this sample?

$X$ is the number of purple balls in this sample

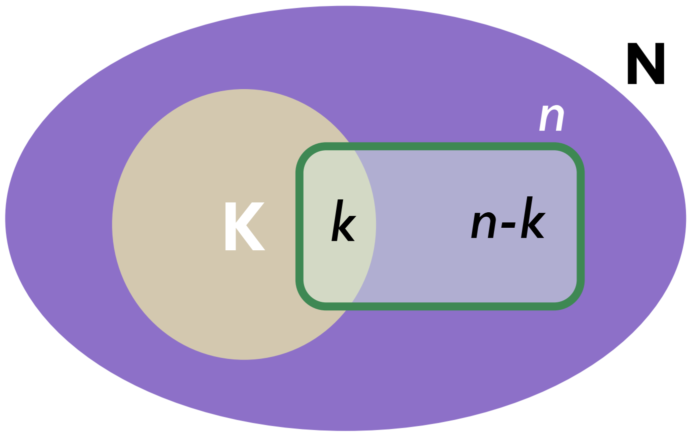
**If you draw out $n$ balls, what is the probability you see $k$ purple ones?**

Of the $K$ purple, we draw out $k$ choose which $k$ will be drawn

Of the $N - K$ other balls, we will draw out $n - k$, choose which $N - K - (n - k)$ will be removed.

Sample space all subsets of size $n$

$$\mathbb{P}(X = k) = \frac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}}$$



# Hypergeometric: Analysis (Expectation)

*$X$ is the number of purple balls in the sample*

$X = D_1 + D_2 + \cdots + D_n$

Where $D_i$ is the indicator that draw $i$ is purple.

$D_1$ is 1 with probability $K/N$.

What about $D_2$?

$$\mathbb{P}(D_2 = 1) = \frac{K-1}{N-1} \cdot \frac{K}{N} + \frac{K}{N-1} \cdot \frac{K-N}{N} = \frac{K(K-N+K-1)}{N(N-1)} = \frac{K}{N}$$

In general $\mathbb{P}(D_i = 1) = \frac{K}{N}$

It might feel counterintuitive, but it's true!

# Hypergeometric: Analysis

$\mathbb{E}[X]$

$= \mathbb{E}[D_1 + \cdots D_n] = \mathbb{E}[D_1] + \cdots + \mathbb{E}[D_n] = n \cdot \frac{K}{N}$

Can we do the same for variance?

No! The $D_i$ are dependent. Even if they have the same probability.

# Hypergeometric Random Variable

$X \sim \text{HypGeo}(N, K, n)$

$X$ is the number of success balls drawn in the sample.

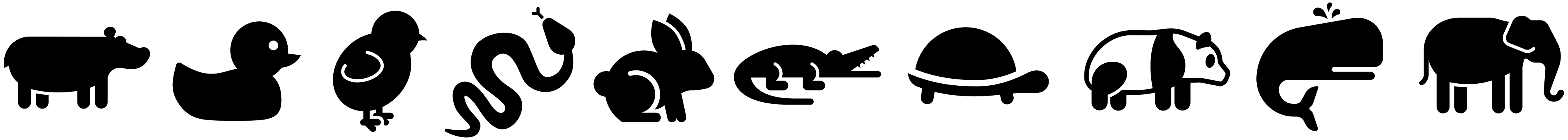
Parameters: A total of $N$ balls in an urn, of which $K$ are successes. Draw $n$ balls without replacement.

**PMF:** $p_X(k) = \frac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}}$

**CDF:** $\mathbb{E}[X] = \frac{nK}{N}$

**Variance:** $\text{Var}(X) = n \cdot \frac{K}{N} \cdot \frac{N-K}{N} \cdot \frac{N-n}{N-1}$

# Zoo!



### $X \sim \mathrm{Unif}(a, b)$

$$p_X(k) = \frac{1}{b-a+1}$$

$$\mathbb{E}[X] = \frac{a+b}{2}$$

$$\mathrm{Var}(X) = \frac{(b-a)(b-a+2)}{12}$$

### $X \sim \mathrm{Ber}(p)$

$$p_X(0) = 1-p; \quad p_X(1) = p$$

$$\mathbb{E}[X] = p$$

$$\mathrm{Var}(X) = p(1-p)$$

### $X \sim \mathrm{Bin}(n, p)$

$$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$\mathbb{E}[X] = np$$

$$\mathrm{Var}(X) = np(1-p)$$

### $X \sim \mathrm{Geo}(p)$

$$p_X(k) = (1-p)^{k-1} p$$

$$\mathbb{E}[X] = \frac{1}{p}$$

$$\mathrm{Var}(X) = \frac{1-p}{p^2}$$

### $X \sim \mathrm{NegBin}(r, p)$

$$p_X(k) = \binom{k-1}{r-1} p^r (1-p)^{k-r}$$

$$\mathbb{E}[X] = \frac{r}{p}$$

$$\mathrm{Var}(X) = \frac{r(1-p)}{p^2}$$

### $X \sim \mathrm{HypGeo}(N, K, n)$

$$p_X(k) = \frac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}}$$

$$\mathbb{E}[X] = n\frac{K}{N}$$

$$\mathrm{Var}(X) = \frac{K(N-K)(N-n)}{N^2(N-1)}$$

### $X \sim \mathrm{Poi}(\lambda)$

$$p_X(k) = \frac{\lambda^k e^{-\lambda}}{k!}$$

$$\mathbb{E}[X] = \lambda$$

$$\mathrm{Var}(X) = \lambda$$

# Zoo Takeaways

You can do relatively complicated counting/probability calculations much more quickly than you could week 1!

You can now explain why your problem is a zoo variable and save explanation on homework (and save yourself calculations in the future).

Don't spend extra effort memorizing…but be careful when looking up Wikipedia articles.

- The exact definitions of the parameters can differ (is a geometric random variable the number of failures before the first success, or the total number of trials including the success?)

# Discrete Zoo of Random Variables

- **Uniform:** Every integer between $a$ and $b$ are equally likely
  $\mathrm{Unif}(a, b)$
- **Bernoulli:** Whether there is success in one trial
  $\mathrm{Ber}(p)$ is 1 with probability $p$ and 0 otherwise
- **Binomial:** Number of successes in $n$ independent trials
  $\mathrm{Bin}(n, p)$ - $n$ independent trials, probability $p$ of success on each trial
- **Geometric:** Number of trials till first success
  $\mathrm{Geo}(p)$ - probability $p$ of success on each trial
- **Poisson:** Number of successes in a time interval
  $\mathrm{Poi}(\lambda)$ - average number of successes in the time interval
- **Negative Binomial**: Number of trials till $r$'th success
  $\mathrm{NegBin}(r, p)$ - probability $p$ of success on each trial, want trials till the $r$'th success
- **Hypergeometric:** Number of successes when drawing a sample
  $\mathrm{HypGeo}(N, K, n)$ - drawing a sample of $n$ items from a set of $N$ with $K$ successes

# Halfway Point!

# What have we done over the past 4 weeks?

Counting

Combinations, permutations, indistinguishable elements, starts and bars, inclusion-exclusion…

Probability foundations

Events, sample space, axioms of probability, expectation, variance

Conditional probability

Conditioning, independence, Bayes' Rule

Refined our intuition

Especially around Bayes' Rule

# What's next?

- Continuous random variables.
  - So far our sample spaces have been countable. What happens if we want to choose a random real number?
  - How do expectation, variance, conditioning, etc. change in this new context?
  - Mostly analogous to discrete cases, but with integrals instead of sums.
- Analysis when it's inconvenient (or impossible) to exactly calculate probabilities.
  - Central Limit Theorem (approximating discrete distributions with continuous ones)
  - Tail Bounds/Concentration (arguing it's unlikely that a random variable is far from its expectation)
- A first taste of making predictions from data (i.e., a bit of ML)