linearity of expectation

LECTURE 9

> **INDEPENDENCE OF RANDOM VARIABLES:**
> X and Y are independent if for all k, ℓ
> $\mathbb{P}(X = k, Y = \ell) = \mathbb{P}(X = k)\mathbb{P}(Y = \ell)$
>
> **EXPECTATION OF A FUNCTION OF X:**
> $\mathbb{E}[g(X)] = \sum_{k \in \Omega_X} g(k) \cdot \mathbb{P}(X = k)$
>
> **LINEARITY OF EXPECTATION:**
> **E[aX+bY+c]=aE[X]+bE[Y]+c**

TODOS FOR WEEK 4

- ☐ Concept check 8 ☐ 9 ☐ 10
- ☐ HW3 due on Wed (7/11) ☐ Coding on Fri. (7/13)
- ☐ Read midterm information
- ☐ Practice for midterm, review session on Friday
- ☐ Attend/participate in section on Thursday

INDEPENDENCE OF RANDOM VARIABLES

"knowing the value of one RV doesn't tell us anything about what the value of the other might be"

X and Y are independent if for all k, ℓ

$$\mathbb{P}(X = k, Y = \ell) = \mathbb{P}(X = k)\mathbb{P}(Y = \ell)$$

Are S="the sum of two dice" and R="the value of the red die" independent?

X ~ "the # of heads in the first n flips." Y ~ "the # of heads in the last n flips." Are X and Y independent?

MUTUAL INDEPENDENCE OF RANDOM VARIABLES

$X_1, X_2, \ldots, X_n$ are mutually independent if for all $x_1, x_2, \ldots, x_n$

$$\mathbb{P}(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n) = \mathbb{P}(X_1 = x_1)\mathbb{P}(X_2 = x_2)\cdots\mathbb{P}(X_n = x_n)$$

MORE OF EXPECTATION!

Expectation formulas:

$$\mathbb{E}[X] = \sum_{k \in \Omega_X} k \cdot \mathbb{P}(X = k) \qquad \mathbb{E}[X] = \sum_{\omega \in \Omega} X(\omega) \cdot \mathbb{P}(\omega)$$

EXPECTATION OF A FUNCTION OF A RANDOM VARIABLE

Functions on random variables are still random variables

e.g.,

$$\mathbb{E}[g(X)] = \sum_{k \in \Omega_X} g(k) \cdot \mathbb{P}(X = k)$$

The probabilities are the same, but we apply the function on the values in the support of X

$$p_X(k) = \begin{cases} 0.5 & k = 1 \\ 0.5 & k = -1 \\ 0 & \text{otherwise} \end{cases} \qquad p_{X^2}(k) =$$

E[X]=__________   E[X^2]=__________

LINEARITY OF EXPECTATION

The expected value of a sum of random variables is the sum of their individual expectations

★**E[X+Y]=E[X]+E[Y]** ★**E[X1+X2+...+Xn]=E[X1]+E[X2]+...+E[Xn]** ★**E[aX+bY+c]=aE[X]+bE[Y]+c**

You catch X fish,with $\mathbb{E}[X]$=3. Your friend catches Y fish, with $\mathbb{E}[Y]$=7.

How many fish do both of you bring on an average day?

› E[_______] =

You can sell each for $10, but you need $15 for expenses. What is your average profit?

› E[_______] =

Oftentimes, the random variables we want to break X down into are indicator random variables

$$X = \begin{cases} 1 & \text{if event A occurs} \\ 0 & \text{otherwise} \end{cases}$$

$$p_X(k) = \begin{cases} \mathbb{P}(A) & \text{if } k = 1 \\ 1 - \mathbb{P}(A) & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} &\mathbb{E}[X] \\ &= 1 \cdot p_X(1) + 0 \cdot p_X(0) \\ &= p_X(1) = \mathbb{P}(A) \end{aligned}$$

GENERAL PROCESS FOR USING LOE WHEN WE HAVE MORE COMPLEX SITUATIONS

› **Decompose**: Decompose the random variable into sum of simple RVs: X=X1+X2+...+Xn

› **LOE**: Apply Linearity of Expectation - E[X]=E[X1]+E[X2]+...+E[Xn]

› **Conquer**: Compute the expectation of each Xi

EXAMPLES...

The probability of flipping a head is p and we flip a coin n times. X ~ total number of heads seen

What is E[X]?

In a class of m students, on average how many pairs of people have the same birthday?

Decompose: Let X be the number of pairs with the same birthday

LOE:

Conquer:

n people are sitting around a circular table. There is a name tag in each place. Nobody is sitting in front of their own name tag. Rotate the table by a random number k of positions between 1 and n-1 (equally likely). Let X be the number of people that end up in front of their own name tag. Find E[X].

Decompose:

LOE:

Conquer: