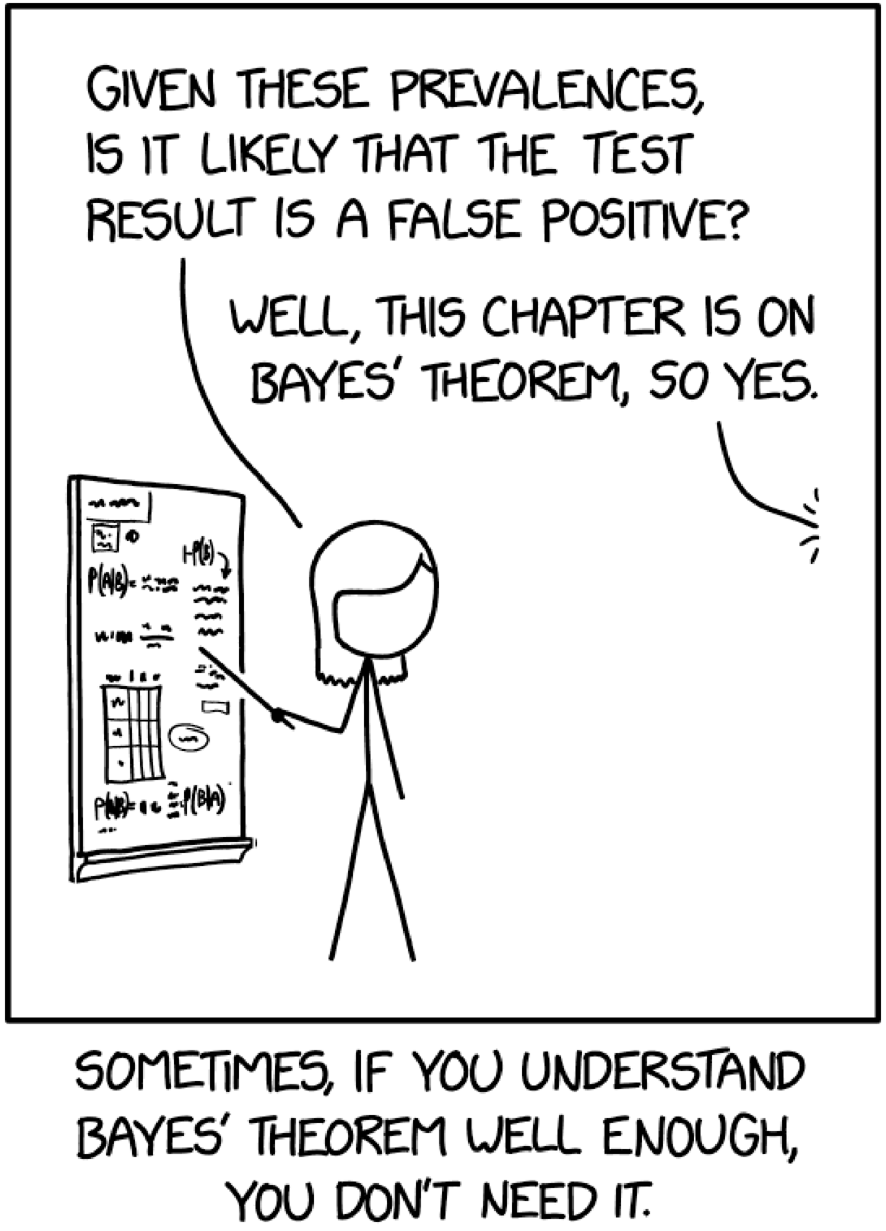

Breather + Review

CSE 312 24Su

Lecture 7

Announcements

HW1 grades are out (regrade request close next Wednesday night)

HW2 due tonight

HW3 out this evening.

> HW3 includes a programming project – using Bayes rule to do some machine learning – detecting whether emails are spam or "ham" (legitimate emails).

> Longer than the programming on HW1 – please get started early!

> Extra resources are available!

No section tomorrow (no section participation for tomorrow)

Section video will be posted walking through some section problems!

No concept check for today's lecture

Outline for Today

> **Combinatorial Proofs**

Reminder of what it is / general tips + a practice proof

> **Counting**

Review of counting techniques

Tips/guidelines for how to figure out what technique to use

Practice problems

> **Uniform Probability Spaces**

How to know to setup a uniform probability spaces

How to setup a *uniform* sample space

Some practice (+ more counting practice)

> **Conditional probability**

Review of what conditional probability/rules we've learned

How to know what rule to use

A practice problem

Combinatorial Proofs

Proof by double counting

Combinatorial Proofs

When to use?
> Algebraic proofs are often difficult and don't give us an "intuitive" reason for why the equation holds
> *Combinatorial proofs* are used for proving an equation that may involve combinations, factorials, permutations, etc.

1. Describe a scenario
2. Explain how the LHS counts the outcomes in that scenario
3. Explain how the RHS counts the outcomes in that <u>same</u> scenario
4. State that "because the LHS and RHS both count the number of outcomes in the same scenario, they must be equal"

Combinatorial Proofs – tips!

> Start with the side that “looks simpler”

> Multiplying terms together indicates some sequential process (the product rule)

> A summation/adding terms indicates some disjoint cases that are added together using the sum rule

Combinatorial Proofs – example

$$\binom{n+2}{3} = \sum_{i=2}^{n+1} (i-1)(n+2-i)$$

Scenario: You have a set $\{1,2,3,\ldots.n+2\}$ and form a <u>subset</u> of 3 distinct elements.

LHS:

RHS:

Combinatorial Proofs – example

$$\binom{n+2}{3} = \sum_{i=2}^{n+1} (i-1)(n+2-i)$$

Scenario: You have a set $\{1,2,3, \ldots .n+2\}$ and form a subset of 3 distinct elements.

LHS: We are picking an unordered subset of 3 distinct elements from a set of $n+2$. We use a combination to count there are $\binom{n+2}{3}$ possible subsets.

RHS:

Combinatorial Proofs – example

$$\binom{n+2}{3} = \sum_{i=2}^{n+1} (i-1)(n+2-i)$$

Scenario: You have a set $\{1,2,3, \dots . n+2\}$ and form a subset or 3 distinct elements.

LHS: We are picking an unordered subset of 3 distinct elements from a set of $n+2$. We use a combination to count there are $\binom{n+2}{3}$ possible subsets.

RHS: We see a summation, which means we are looking at some disjoint cases. In each case we still want to pick 3 elements. Let's think about the case $n = 10, i = 3$

In this case, the formula says we have $2 \cdot 9$ options.

1 2 3 4 5 6 7 8 9 10 11 12

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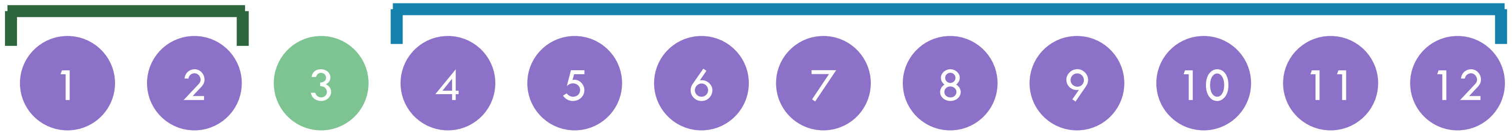

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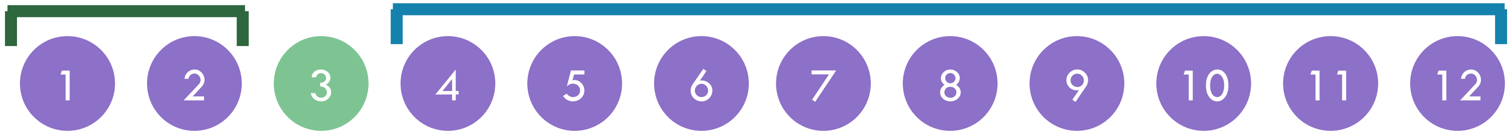
In this case, the formula says we have $\mathbf{2} \cdot \mathbf{9}$ options.

So… our cases are based on what the "middle" is. Then we pick the "left" and "right".



Combinatorial Proofs – example

$$\binom{n+2}{3} = \sum_{i=2}^{n+1} (i-1)(n+2-i)$$

Scenario: You have a set $\{1,2,3, \ldots .n+2\}$ and form a subset or 3 distinct elements.

LHS: We are picking an unordered subset of 3 distinct elements from a set of $n+2$. We use a combination to count there are $\binom{n+2}{3}$ possible subsets.

RHS: We are looking at cases based on what the middle value in order in the subset is. These cases go from i=2 to n+1. For each possible "middle value" we need to pick a value smaller than i and one larger than i to create this set of 3. There are $i-1$ values smaller than i and ≥ 1. There are $n+2-i$ values larger than i - between i (exclusive) and $n+2$. So, there are $(i-1)(n+2-i)$ ways to pick to remaining two values (we use the product rule because we are using a sequential process where we first pick the smaller, then the larger number. We sum these together because these are disjoint cases.

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RHS: We are looking at cases based on what the middle value in order in the subset is – the possible values of the middle values is between 2 and $n+1$. For each possible "middle value" we need to pick a value smaller than i and one larger than i to create this set of 3. There are $i-1$ values smaller than i and ≥ 1. There are $n+2-i$ values larger than i - between i (exclusive) and $n+2$. So, there are $(i-1)(n+2-i)$ ways to pick to remaining two values (we use the product rule because we are using a sequential process where we first pick the smaller, then the larger number. We sum these together because these are disjoint cases.

Since both sides count the number of options in the same scenario, they are equal.

Counting

Counting the number of ways to do something,
the size of a set

Lots of counting…

Factorial: $n!$ ways to rearrange n distinct things

Complementary counting: Counting the ways for A to not occur
$\text{ways for A to NOT occur} = \text{total options} - \text{ways for A to occur}$

Product rule: *Sequential process* with m_1 options in 1st step, m_2 options in 2nd , m_3 in 3rd step, etc. we pick 1 option from each to form the outcome $m_1 \cdot m_2 \cdot m_3 \cdot \ldots$

Stars and bars: $\binom{n+k-1}{k-1}$ ways to distribute n identical things to k distinct types

Picking k distinct elements from a group of n distinct elements

Permutations: $P(n, k)$ if the **order of the k elements does matter**

Combinations: $\binom{n}{k}$ if the **order of the k elements does not matter**

Finding the size of a union of sets - $|A \cup B \cup \cdots|$

Sum rule: If disjoint, $|A| + |B| + \cdots$

Inclusion-Exclusion: *singles-doubles+triples-…*

How to tell which approach(es) to take?

- Identify keywords in the problem, and key properties, and think about what techniques match those patterns
- If an approach isn't working or things are getting out of control, try a different approach!
- There are often multiple ways to solve the same problem ☺
- In some problems, we might need to start by solving a simplified version of the problem and then dividing/subtracting out overcounting

This all takes practice so don't be hard on yourself if you don't think of the correct answer at first glance!!

How to tell which approach(es) to take?

> Can we break this into some disjoint cases? (**sum rule**)

> Can we create our desired outcomes with a clear sequential process? (**product rule**)

> Are we counting the ways for something <u>not</u> to occur? (**complementary counting**)

> Are we dealing with the union of some sets? (**inclusion-exclusion**)

> Does order matter or not in this problem? (**permutation** or **combination**)

> Are the objects distinguishable or indistinguishable (**stars and bars** if indistinguishable)?

How to tell which approach(es) to take?

"at least one of A, B, <u>or</u> C", "either A <u>or</u> B <u>or</u> C" -> $|A \cup B \cup C|$

sum rule *if disjoint* o.w. **inclusion-exclusion** OR **take complement** to find none A, B, C)

"<u>none of</u> A, B, C" -> $|\overline{A} \cap \overline{B} \cap \overline{C}|$

take the complement and find the above OR you might be able to **setup a sequential process** in a way that ensures none of these happen by restricting certain options

"items of same type are <u>indistinguishable</u>", "only care about <u>how many of each type we select</u>", "distribute <u>identical</u> into distinguishable bins"

use the **stars and bars** approach with the identical items as stars and bins/types as bins

Sleuth's Criterion

How to check if we counted correctly?

For each outcome that we want to count, there should be exactly one set of choices in the sequential process that will lead to that outcome.

> If there are no sequence of choices that will lead to the outcome, we have **undercounted**.

> If there is more than one sequence of choices that will lead to the outcome, we have **overcounted**.

Try writing down some possible outcomes, and make sure that there's exactly 1 way to create each outcome with your sequential process!

We use permutation or combination when selecting <u>distinct</u> elements from a group of <u>distinct</u> elements

But how to tell if order matters?

Try listing out some possible outcomes and think about, what outcomes do I want to be counted the same and what outcomes do I want to be counted separately?

For example....

> **Counting paths from (0,0) to (5,3):** Picking 3 of 8 steps to go UP
> {S4, S2, S7} corresponds to the same path as {S2, S7, S4} -> **combination**

> **Picking 4 people to be on a team and assign them soccer positions**
> {Position 1: A, Position 2: B, Position 3: C, Position 4: D}, {Position 1: B, Position 2: D, Position 3: C, Position 4: A}, etc. are all corresponding to same subset but different positions/outcomes, so order does matter here -> **permutation**

We use permutation or combination when selecting <u>distinct</u> elements from a group of <u>distinct</u> elements

But how to tell if order matters?

For example, in the birthday problem from last week…

We ended up need to find $|\bar{E}|$ *which is the number of possible birthday assignments so that everyone has a different birthday.*

We need to pick 50 **distinct** birthdays from a group of 365 **distinct** b-days

So….we're going to use a *permutation* or a *combination*. Which one?

We use permutation or combination when selecting <u>distinct</u> elements from a group of <u>distinct</u> elements

3

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So….we're going to use a *permutation* or a *combination.*

Should {July 1st, September 28, December 4^{th} …} be counted only once?

No! That won't count for how those can be assigned to the people.

So, the order DOES matter -> permutation

Example

A company has 10 projects and 15 employees. Each project requires a team of 3 employees. How many ways can you assign employees to projects if each employee can be part of multiple projects, but no projects can have the same team?

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Our first approach might be to use complementary counting…

What would the complement be in this problem?

> *"at least two projects with the same team"* – this seems hard to count

Example

A company has 10 projects and 15 employees. Each project requires a team of 3 employees. How many ways can you assign employees to projects if each employee can be part of multiple projects, but no projects can have the same team?

Let's rephrase the problem…

"How many options if all projects have a distinct team of 3 people"

So, we are selecting **10 <u>distinct</u> teams of 3** from **all possible <u>distinct</u> teams**

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> How many possible teams of 3 are there?

> How many ways are to pick teams of 3 for these 10 projects?

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> **How many possible teams of 3 are there?** $\binom{15}{3} = 455$

Picking a *subset* of 3 distinct people from the 15 employees -> combination

> **How many ways are to pick distinct teams for these 10 projects?**

Example

A company has 10 projects and 15 employees. Each project requires a team of 3 employees. How many ways can you assign employees to projects if each employee can be part of multiple projects, but no projects can have the same team?

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> **How many possible teams of 3 are there?** $\binom{15}{3} = 455$
>
> Picking a *subset* of 3 distinct people from the 15 employees -> combination
>
> **How many ways are to pick teams of 3 for these 10 projects?** $P(455, 10)$
>
> Picking 10 of these teams *and* assigning them to project -> permutation

Example

A company has 10 projects and 15 employees. Each project requires a team of 3 employees. How many ways can you assign employees to projects if each employee can be part of multiple projects, but no projects can have the same team?

Another valid approach!

Step 1, pick for 1st project: $\binom{15}{3}$

Step 2, pick for 2nd project: $\binom{15}{3} - 1$

Step 3, pick for 3nd project: $\binom{15}{3} - 2$

…

Step 10, pick for 10th project: $\binom{15}{3} - 9$

Use <u>product rule</u>, and multiply all these together!

What to do when we see "at least 1…"'?

It depends!! Here are some things to try:

- Taking the **complement**. Sometime the complement of what we're trying to count is easier, so we use <u>complementary counting</u>.
  - e.g., the complement of at least 1 vowel in the string is strings with NO vowels
- Using **casework**. If the scale of the problem isn't too big, we can look at each case of exactly how many things are included – then we <u>sum rule</u>
  - e.g., we can look at the case of exactly 1 vowel, exactly 2 vowels, exactly 3, etc.
- Write it as a **union** of sets and use the <u>sum rule</u> if disjoint and <u>inclusion-exclusion</u> if they overlap
- Write a **sequential process** and use the product rule
  - Especially with 'at least __' problems it's very easy to overcount, so either *divide out overcounting* or make sure to set up the sequential process in a way that overcounts
- **If we're dealing with indistinguishable things**, and using stars and bars and want to have at least k of some types….

Start by just taking k of that type (only 1 option) and then using stars and bars for the remaining

For example....remember this problem?

You're organizing a potluck party with dishes from 4 different cuisines: Italian, Thai, Lebanese, and Nigerian. Each cuisine has 10 distinct dishes available. How many ways can you select a plate of 5 distinct dishes to taste, ensuring that you try **at least one dish from each cuisine**? The order of dishes doesn't matter.

Let's first try...taking the complement!

What's the complement of "at least one dish from each cuisine"?

> "at least one cuisine with no dishes" *which means that...*

> "I. has no dishes **OR** T. has no dishes **OR** L. has no dishes **OR** N. has no dishes"

> the "**OR**" means a union of sets

--- do these sets overlap? **Yes**, so we would have to use inlusion-exclusion

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That was too much work ☹ Let's now try…using casework!

If we write out some possibilities:

$\{I_1, I_5, T_6, L_9, N_3\}, \{I_1, T_5, T_6, L_9, N_3\}, \{I_1, T_5, L_6, L_9, N_3\}, \{I_1, T_5, L_6, N_9, N_3\}$

We see that we have the cases:

> Case 1 (exactly 2 of Italian and exactly 1 of 3 remaining): $\binom{10}{2} \cdot \mathbf{10} \cdot \mathbf{10} \cdot \mathbf{10}$

> Case 2 (exactly 2 of Thai and exactly 1 of 3 remaining): $\binom{10}{2} \cdot \mathbf{10} \cdot \mathbf{10} \cdot \mathbf{10}$

> Case 3 (exactly 2 of Lebanese and exactly 1 of 3 remaining): $\binom{10}{2} \cdot \mathbf{10} \cdot \mathbf{10} \cdot \mathbf{10}$

> Case 4 (exactly 2 of Nigerian and exactly 1 of 3 remaining): $\binom{10}{2} \cdot \mathbf{10} \cdot \mathbf{10} \cdot \mathbf{10}$

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If we didn't think of that...we might now try writing as union of sets

It's not really clear how we could write this as a union of sets. If we expanded this out, we would be looking for:

"at least one dish from I **AND** at least one dish from T **AND** ...."

This is an intersection – not a union, which is why when we took the complement it was a union

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Lastly…we will try writing a sequential process!

Approach 1:

We want at least 1 dish in each cuisine, so let's start by "forcing" that:

Step 1: Pick 1 Italian dish – 10 options

Step 2: Pick 1 Thai dish – 10 options

Step 3: Pick 1 Lebanese dish – 10 options

Step 4: Pick 1 Nigerian dish – 10 options

Step 5: Pick a 5th dish from remaining: $40 - 4 = 36$ options

$10^4 \cdot 36$ options

Let's check that it doesn't overcount….

For example….remember this problem?

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Any set of dishes has last dish a repeat cuisine:

$\{I_a, T_b, L_c, N_d, T_e\}$

Can be made by picking T_b in step 2 and T_e in step 5 OR picking T_e in step 2 and T_b in step 5

For example….remember this problem?

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Step 4: Pick 1 Nigerian dish – 10 options

Step 5: Pick a 5th dish from remaining : $40 - 4 = 36$ options

Divide by 2 because every possible outcome is counted twice: $(10^4 \cdot 36)/2$

For example….remember this problem?

You're organizing a potluck party with dishes from 4 different cuisines: Italian, Thai, Lebanese, and Nigerian. Each cuisine has 10 distinct dishes available. How many ways can you select a plate of 5 distinct dishes to taste, ensuring that you try **at least one dish from each cuisine**? The order of dishes doesn't matter.

Lastly…we will try writing a sequential process!

Approach 2:

When we try listing out some possibilities, there's always 1 repeated cuisine

Step 1: Pick 1 cuisine that will be repeated – 4 options

Step 2: Pick 2 dishes from that cuisine – $\binom{10}{2}$ options

Step 3: Pick 1 dish from the 1st remaining cuisine – 10 options

Step 4: Pick 1 dish from the 2st remaining cuisine – 10 options

Step 5: Pick 1 dish from the 3rd remaining cuisine – 10 options

$\{I_a, T_b, L_c, N_d, T_e\}$
Can be made with only set of choices, so **no** overcounting

A Fruit Problem

You have to choose 8 pieces of fruit. There are apples, oranges, and bananas. Fruits of the same type are indistinguishable.

You need to pick **at most** 2 apples and **at least** 1 banana. How many sets of fruit can you choose?

When you see a restriction with **at most $\boldsymbol{k}$** of something either use ***casework*** if k is small enough OR ***take complement*** and find total – was with **more** than k of that thing (very similar to *at least* k of something)

A Fruit Problem

You have to choose 20 pieces of fruit. There are apples 🍎, oranges 🍊, bananas 🍌, and cherries 🍒. Fruits of the same type are indistinguishable. You need to pick **at most** 2 apples and **at least** 1 banana. How many sets of fruit can you choose?

Divide into <u>disjoint cases</u> based on number of apples:

0 apples:

1 apple 🍎:

2 apples 🍎 🍎:

______________ total (by sum rule)

A Fruit Problem

You have to choose 20 pieces of fruit. There are apples 🍎, oranges 🍊, bananas 🍌, and cherries 🍒. Fruits of the same type are indistinguishable. You need to pick **at most** 2 apples and **at least** 1 banana. How many sets of fruit can you choose?

Divide into <u>**disjoint cases**</u> based on number of apples:

0 apples: Pick 1 🍌, select 🍊 / 🍌 / 🍒 for remaining 19: $\binom{19+3-1}{3-1} = \binom{21}{2}$

1 apple 🍎: Pick 1 🍌, select 🍊 / 🍌 / 🍒 for remaining 18: $\binom{18+3-1}{3-1} = \binom{20}{2}$

2 apples 🍎 🍎: Pick 1 🍌, select 🍊 / 🍌 / 🍒 for remaining 19: $\binom{17+3-1}{3-1} = \binom{19}{2}$

$\binom{21}{2} + \binom{20}{2} + \binom{19}{2}$ total (by sum rule)

A Fruit Problem (*another* approach!)

You have to choose 20 pieces of fruit. There are apples 🍎, oranges 🍊, bananas 🍌, and cherries 🍒. Fruits of the same type are indistinguishable. You need to pick **at most** 2 apples and **at least** 1 banana. How many sets of fruit can you choose?

1. Pick out your first banana

2. Pick 19 fruits (≤ 2 apples)
   Complementary counting:
   Total possibilities ignoring apple restriction:
   Possibilities with ≥ 3 apples (these are the outcomes we want to **exclude**):

Total: $\binom{22}{3} - \binom{19}{3}$

A Fruit Problem (*another* approach!)

You have to choose 20 pieces of fruit. There are apples 🍎, oranges 🍊, bananas 🍌, and cherries 🍒. Fruits of the same type are indistinguishable. You need to pick **at most** 2 apples and **at least** 1 banana. How many sets of fruit can you choose?

1. Pick out your first banana – 1 option because indistinguishable

2. Pick 19 fruits (≤ 2 apples)

   Complementary counting:

   Total possibilities ignoring apple restriction: $\binom{19+4-1}{4-1}$ (19 pieces, 4 types)

   Possibilities with ≥ 3 apples (these are the outcomes we want to **exclude**)

   $\binom{16+4-1}{4-1}$. Pick 3 apples (1 options bc indistinguishable), pick 4 from 3 types

Total: $\binom{22}{3} - \binom{19}{3}$

Binomial Theorem

Binomial theorem

The Binomial Theorem

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$$

Understand where this formula is coming from, and how to use this to find coefficient of terms

Uniform Probability Spaces

Probability (*in general*)

Probability allows us to assign a value between 0 and 1 to outcomes

- ***Random experiment:*** any process where the outcome is unknown
- ***Sample space*** of an experiment: *set* of all possible outcomes
- ***Event:*** a subset of the sample space (some set of outcomes)
- ***Probability space:*** the pair $(\Omega, \mathbb{P})$ where Ω is the sample space and $\mathbb{P}$ is the probability measure (a *function* that assigns probabilities to every outcome ω in the sample space)

Uniform probability space: A probability space where *every outcome is equally likely to occur*. In a uniform probability space…

$$\mathbb{P}(\omega) = \frac{1}{|\Omega|}, \ \mathbb{P}(E) = \frac{|E|}{|\Omega|}$$

When do I know to setup a uniform probability space?

If you're looking for a probability and see phrases like "<u>equally likely</u>", "<u>uniformly at random</u>", a "<u>fair</u> coin/dice", "<u>randomly</u> picking…" you're most likely going to be able to pick a uniform probability space.

> Pick a sample space where every outcome is **equally likely**
the information included in the sample space should be what's relevant to the problem
think about what are you considering as 1 outcome, write out some examples

> Find the size of the sample space (using counting techniques!)

> Define the event and count its size (using counting techniques!)

> Find the probability by doing $\mathbb{P}(E) = \frac{|E|}{|\Omega|}$

Examples

What is the probability that when you pick 3 distinct cards from a deck of 52 cards, that they form a consecutive sequence (any order)?
Assume you are equally likely to draw any cards.

Sample space:

Event:

Probability:

Examples

What is the probability that when you pick 3 distinct cards from a deck of 52 cards, that they form a consecutive sequence (any order)?
Assume you are equally likely to draw any cards.

Sample space: Set of all possible subsets of 3 cards. $|\Omega| = \binom{52}{3}$

Event: $\boldsymbol{E}$ is the event that the 3 cards form a sequence

Probability: $\mathbb{P}(E) = \frac{|E|}{|\Omega|} = \frac{|E|}{\binom{52}{3}}$

Conditional Probability

What is conditional probability?

$\mathbb{P}(A|B)$ is the probability that A happens <u>given/conditioning on</u> B

we are looking for the probability that A happens but we are given the extra information that B happens

"what's the probability of A <u>given</u> B"

"<u>if</u> B, what's the probability A happens"

e.g., "*What is the probability* ***the moon landing is fake*** *<u>if I wear a tinfoil hat</u>?*"

Let the event $M \sim$ the moon landing is fake

Let the event $T \sim$ I wear a tinfoil hat

We are interested in $\mathbb{P}(M|T)$

Rules that use conditional probability

Definition of conditional probability: $\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$

Bayes' theorem: $\mathbb{P}(B|A) = \frac{\mathbb{P}(A|B)\mathbb{P}(B)}{\mathbb{P}(B)}$

Law of total probability:

$\mathbb{P}(A) = \mathbb{P}(A|E_1)\mathbb{P}(E_1) + \cdots + \mathbb{P}(A|E_n)\mathbb{P}(E_n)$
if $E_1, E_2, \ldots, E_n$ partition the sample space Ω

Chain Rule:
$\mathbb{P}(E_1 \cap E_2 \cap \cdots \cap E_n) = \mathbb{P}(E_1)\mathbb{P}(E_2|E_1)\mathbb{P}(\mathrm{E}_3|\mathrm{E}_1 \cap E_2) \ldots \mathbb{P}(E_n|E_1 \cap \cdots \cap E_{n-1})$

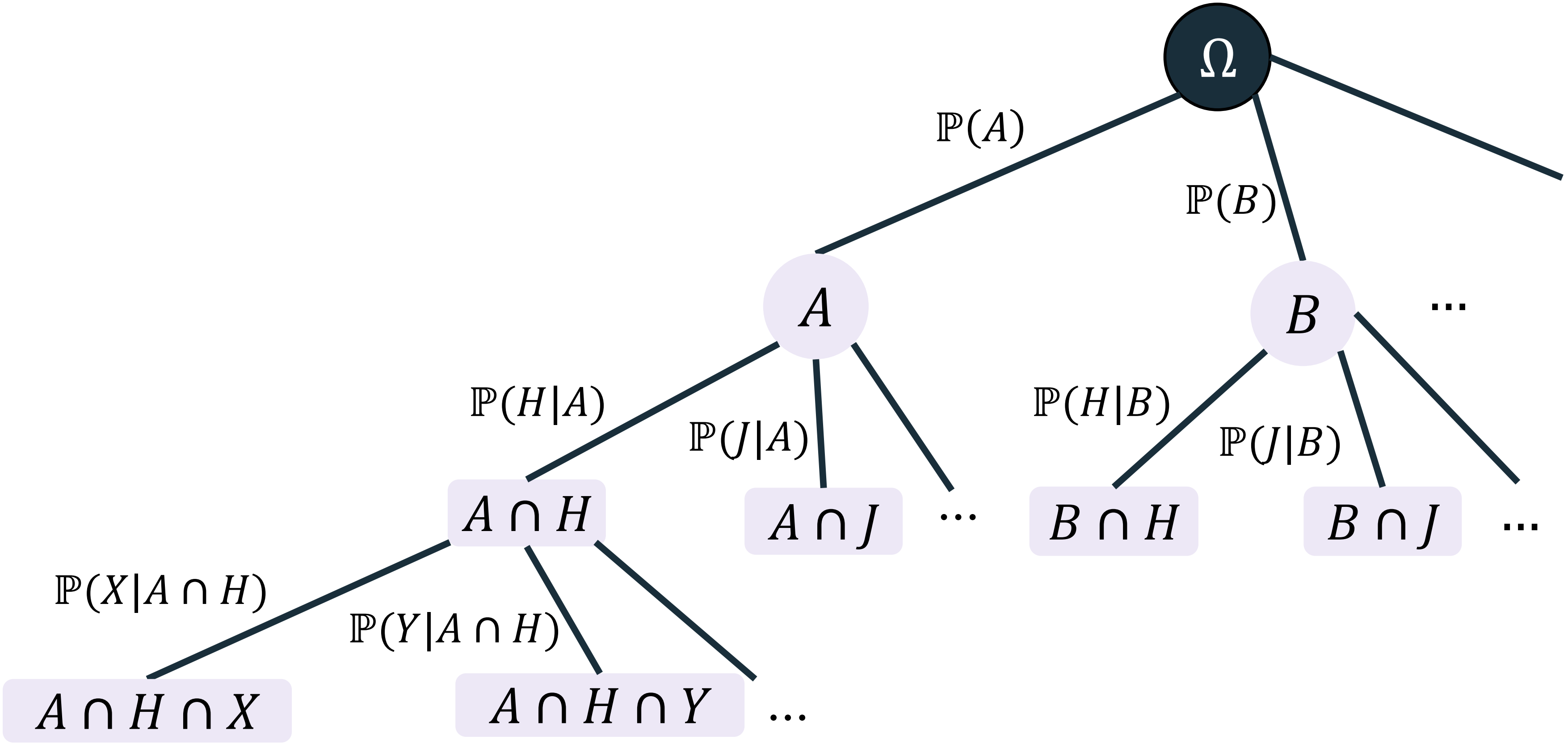
Rules that use conditional probability

If events $\boldsymbol{A}$ and $\boldsymbol{B}$ are *independent*: $\mathbb{P}(A|B) = \mathbb{P}(B), \mathbb{P}(\mathrm{A} \cap B)\mathbb{P}(A)\mathbb{P}(B)$

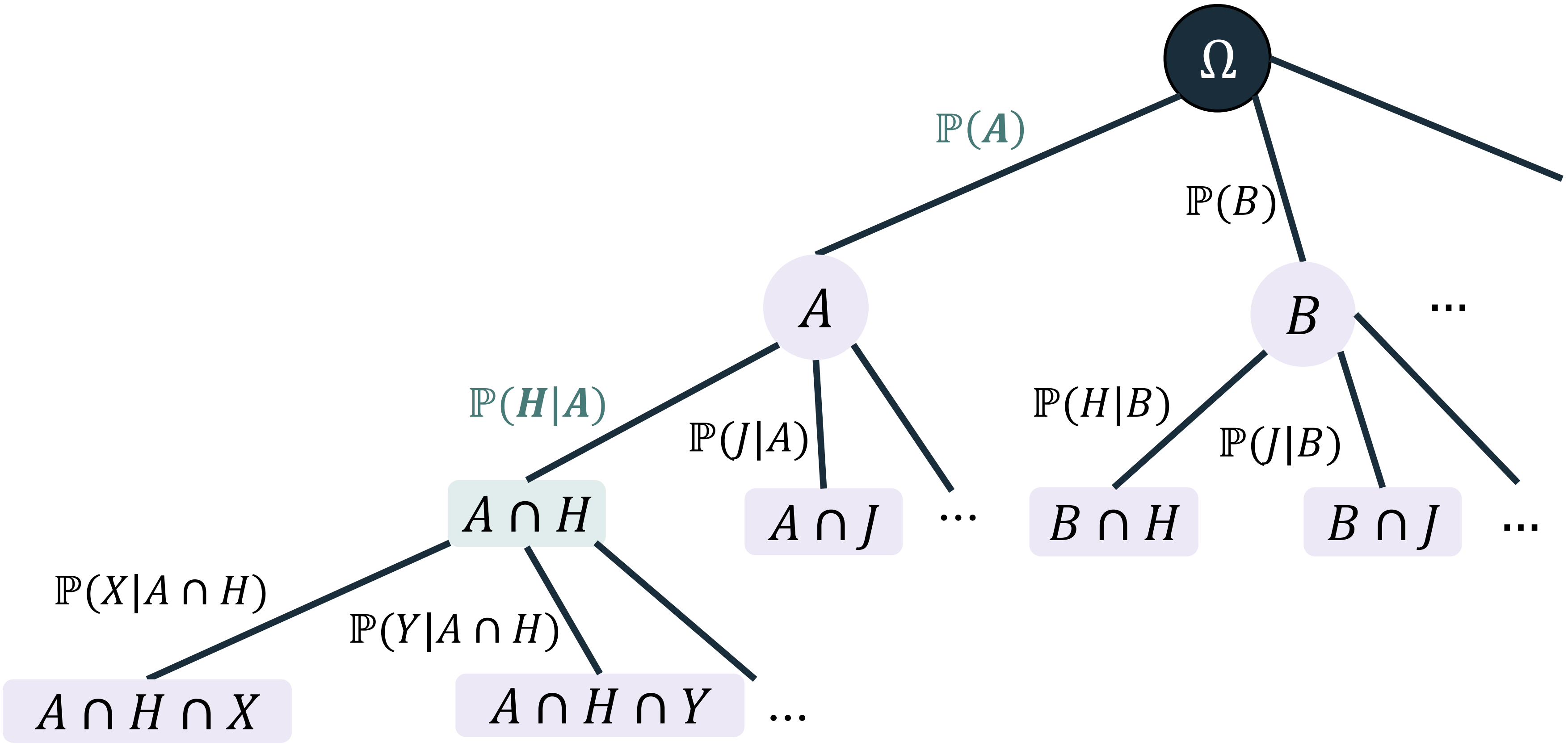
If events $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \ldots$ are *mutually independent*: $\mathbb{P}(\mathrm{A} \cap B \cap C \cap \cdots) = \mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C) \ldots$

If events $\boldsymbol{A}$ and $\boldsymbol{B}$ are *conditionally independent* on $\boldsymbol{C}$: $\mathbb{P}(A \cap B|C) = \mathbb{P}(A|C)\mathbb{P}(B|C)$

A Visual for the Chain Rule (and LoTP!)



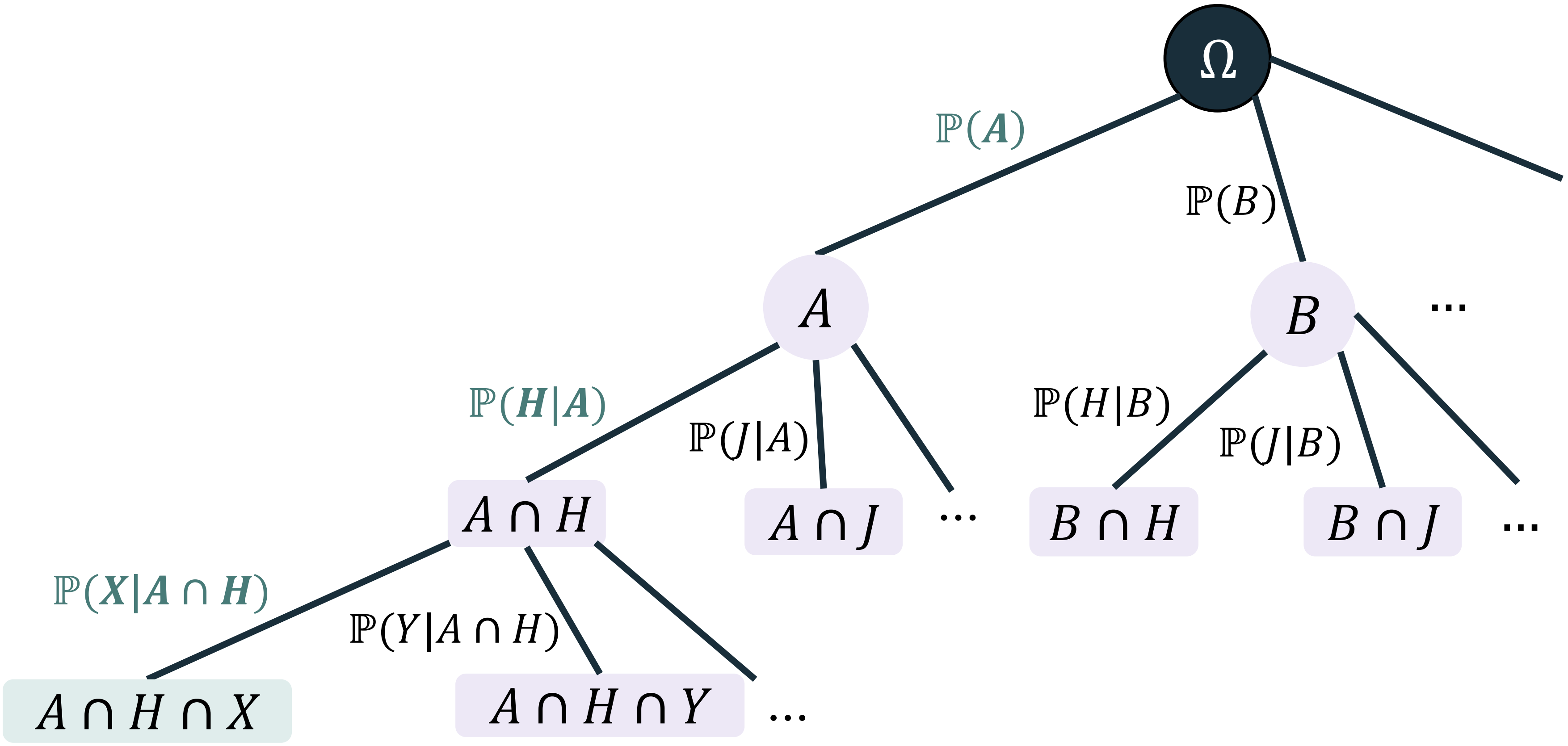
A Visual for the Chain Rule (and LoTP!)



Chain Rule: The probability of one of the "nodes" (e.g.,$\mathbb{P}(A \cap H)$) is the product of the probabilities of each of the steps that led to it:

$$\mathbb{P}(A \cap H) = \mathbb{P}(A)\mathbb{P}(H|A)$$

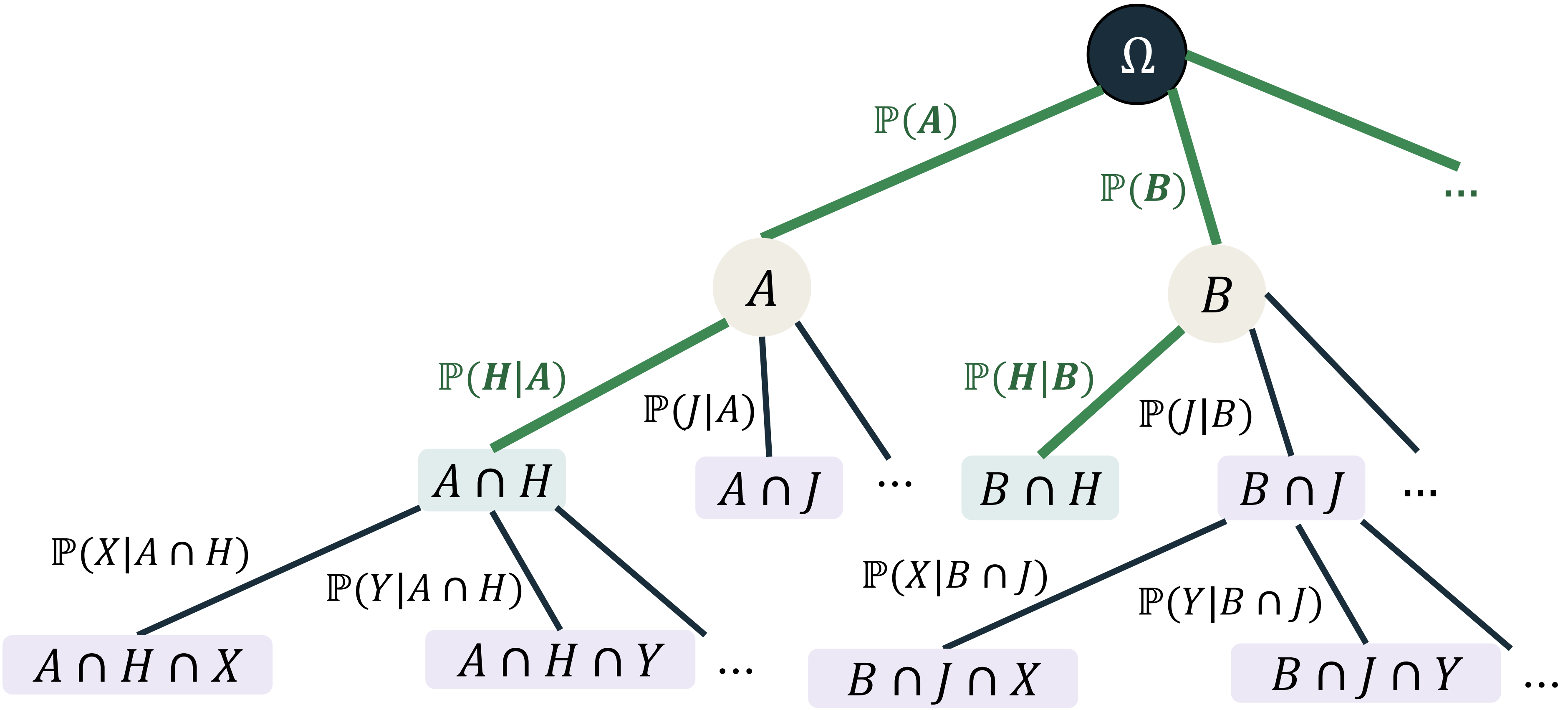
A Visual for the Chain Rule (and LoTP!)



Chain Rule: The probability of one of the "nodes" (e.g.,$\mathbb{P}(A \cap H \cap X)$) is the product of the probabilities of each of the steps that led to it:

$$\mathbb{P}(A \cap H \cap X) = \mathbb{P}(A \cap H)\mathbb{P}(X|A \cap H) = \mathbb{P}(A)\mathbb{P}(H|A)\mathbb{P}(X|A \cap H)$$

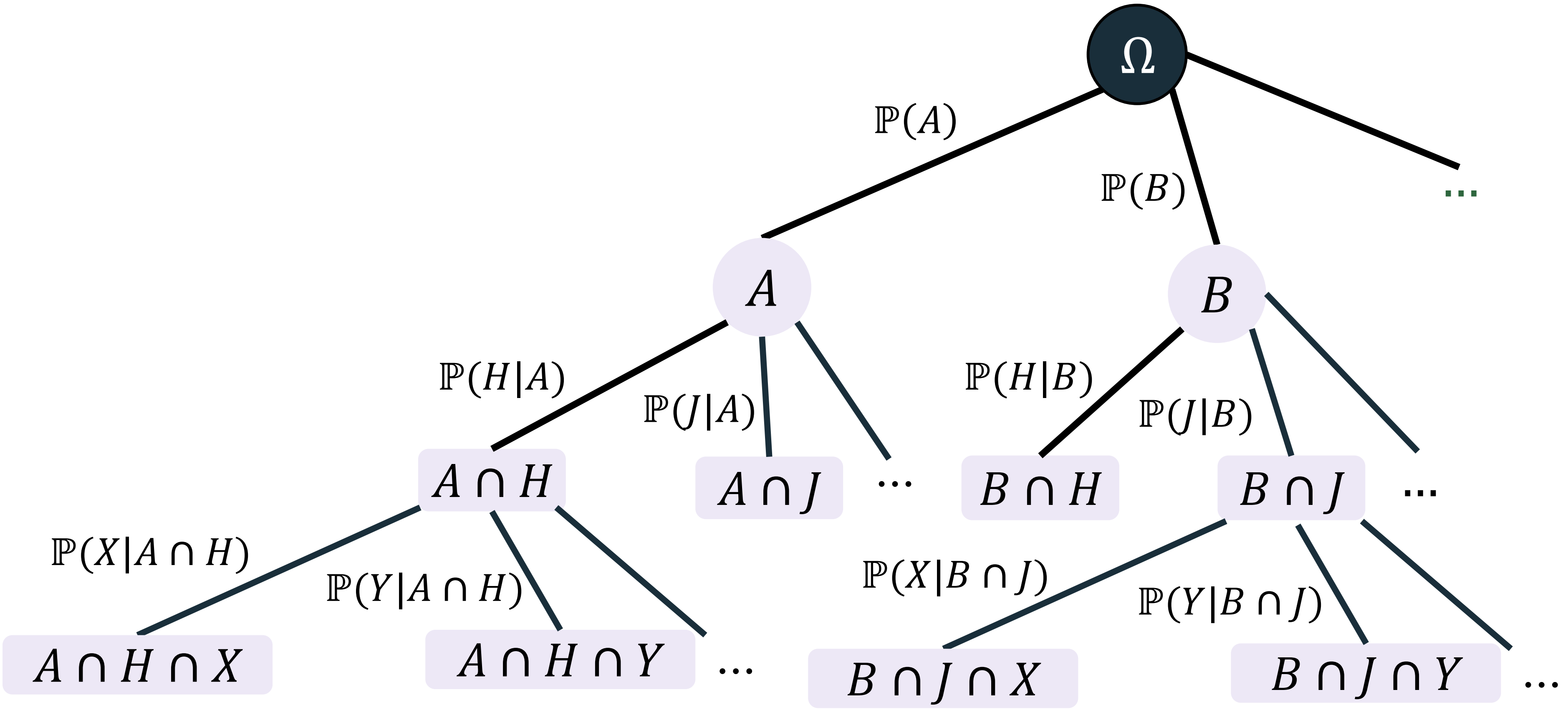
A Visual for the Chain Rule (and LoTP!)



Law of Total Probability: Sum up probabilities of intersection with other events

$$\mathbb{P}(H) = \mathbb{P}(A \cap H) + \mathbb{P}(B \cap H) + \cdots = \mathbb{P}(A)\mathbb{P}(H|A) + \mathbb{P}(B)\mathbb{P}(H|B)$$

A Visual for the Chain Rule (and LoTP!)



⚠️ on homeworks, you can use these kinds of visuals to help with reasoning, but make sure to still explicitly show applications of chain rule, LoTP, etc.) ⚠️

How to know what rules to use?

- Start by clearly defining relevant events for the problem
- Write the probabilities given in the problem in terms of those events
- Write the probability we are looking for in terms of those events
- Think about what rules we can use to relate those probabilities together to get what we are looking for
  - e.g., if we're looking for $\mathbb{P}(A)$ and we have the probabilities for A conditioned on other things, use law of total probability

Example

A classifier determines whether an email is spam.

- The classifier correctly classifies spam emails with probability 0.95 and incorrectly classifies non-spam emails as spam with probability 0.05

The overall probability that an email is spam is 0.3

If an email is classified as spam, what is the probability that it is actually spam?

A classifier determines whether an email is spam.

- The classifier correctly classifies spam emails with probability 0.95 and incorrectly classifies non-spam emails as spam with probability 0.05

The overall probability that an email is spam is 0.3

If an email is classified as spam, what is the probability that it is actually spam?

Start by defining events:

C ~ classifier labels email as spam, S~the email is actually spam

Write probabilities in the problem in terms of events:

The classifier correctly classifies spam emails with probability $\mathbf{0.95}$ and incorrectly classifies non-spam emails as spam with probability $\mathbf{0.05}$

The overall probability that an email is spam is $\mathbf{0.3}$

Start by defining events:

A ~ classifier labels email as spam, $S \sim$the email is actually spam

Write probabilities in the problem in terms of events:

The classifier correctly classifies spam emails with probability $\mathbf{0.95}$ and incorrectly classifies non-spam emails as spam with probability $\mathbf{0.05}$

The overall probability that an email is spam is $\mathbf{0.3}$

Start by defining events:

A ~ classifier labels email as spam, S~the email is actually spam

Write probabilities in the problem in terms of events:

"Classifier correctly classifies spam emails with probability $\mathbf{0.9}$*"* can be reworded as *"If an email is spam, classifier correctly classifies it with probability* $\mathbf{0.9}$*"*

The classifier correctly classifies spam emails with probability $\mathbf{0.95}$ and incorrectly classifies non-spam emails as spam with probability 0.05

The overall probability that an email is spam is 0.3

Start by defining events:

A ~ classifier labels email as spam, S~the email is actually spam

Write probabilities in the problem in terms of events:

"Classifier correctly classifies spam emails with probability 0.9" can be reworded as "*If an email is spam, classifier correctly classifies it with probability* 0.9" → $\mathbb{P}(A|S) = 0.95$

The classifier correctly classifies spam emails with probability 0.95 and incorrectly **classifies non-spam emails as spam with probability $\mathbf{0.05}$**

The overall probability that an email is spam is 0.3

Start by defining events:

A ~ classifier labels email as spam, S~the email is actually spam

Write probabilities in the problem in terms of events:

"*Classifier correctly classifies spam emails with probability* 0.9" can be reworded as "*If an email is spam, classifier correctly classifies it with probability* 0.9" → $\mathbb{P}(A|S) = 0.95$

The classifier correctly classifies spam emails with probability 0.95 and incorrectly **classifies non-spam emails as spam with probability $\mathbf{0.05}$**

The overall probability that an email is spam is 0.3

Start by defining events:

A ~ classifier labels email as spam, S~the email is actually spam

Write probabilities in the problem in terms of events:

"Classifier correctly classifies spam emails with probability 0.9*"* can be reworded as *"If an email is spam, classifier correctly classifies it with probability* 0.9*"* → $\mathbb{P}(A|S) = 0.95$

"Classifier classifies non-spam emails as spam with probability 0.05" can be reworded as "If an email is non-spam, classifies it as spam with probability 0.05*"* → $\mathbb{P}(A|\overline{S}) = 0.05$

The classifier correctly classifies spam emails with probability 0.95 and incorrectly classifies non-spam emails as spam with probability 0.05

The overall probability that an email is spam is $\mathbf{0.3}$

Start by defining events:

A ~ classifier labels email as spam, S~the email is actually spam

Write probabilities in the problem in terms of events:

$\mathbb{P}(A|S) = 0.95$, $\mathbb{P}(A|\overline{S}) = 0.05$

The classifier correctly classifies spam emails with probability 0.95 and incorrectly classifies non-spam emails as spam with probability 0.05

The overall probability that an email is spam is $\mathbf{0.3}$

Start by defining events:

A ~ classifier labels email as spam, $S \sim$the email is actually spam

Write probabilities in the problem in terms of events:

$\mathbb{P}(A|S) = 0.95$, $\mathbb{P}\left(A|\overline{S}\right) = 0.05$, $\mathbb{P}(\boldsymbol{S}) = \mathbf{0.3}$

The classifier correctly classifies spam emails with probability 0.95 and incorrectly classifies non-spam emails as spam with probability 0.05

The overall probability that an email is spam is 0.3

If an email is classified as spam, what is the probability that it is actually spam?

Start by defining events:

A ~ classifier labels email as spam, S~the email is actually spam

Write probabilities in the problem in terms of events:

$\mathbb{P}(A|S) = 0.95$, $\mathbb{P}(A|\overline{S}) = 0.05$, $\mathbb{P}(S) = 0.3$

Write the probability of what we are looking for:

The classifier correctly classifies spam emails with probability 0.95 and incorrectly classifies non-spam emails as spam with probability 0.05

The overall probability that an email is spam is 0.3

If an email is classified as spam, what is the probability that it is actually spam?

Start by defining events:

A ~ classifier labels email as spam, S~the email is actually spam

Write probabilities in the problem in terms of events:

$\mathbb{P}(A|S) = 0.95$, $\mathbb{P}(A|\overline{S}) = 0.05$, $\mathbb{P}(S) = 0.3$

Write the probability of what we are looking for:
$\mathbb{P}(S|A) =$

The classifier correctly classifies spam emails with probability 0.95 and incorrectly classifies non-spam emails as spam with probability 0.05

The overall probability that an email is spam is 0.3

If an email is classified as spam, what is the probability that it is actually spam?

Start by defining events:

A ~ classifier labels email as spam, S~the email is actually spam

Write probabilities in the problem in terms of events:

$\mathbb{P}(A|S) = 0.95$, $\mathbb{P}(A|\overline{S}) = 0.05$, $\mathbb{P}(S) = 0.3$

Write the probability of what we are looking for:

$$\mathbb{P}(S|A) = \frac{\mathbb{P}(A|S)\mathbb{P}(S)}{\mathbb{P}(A)} =$$

The classifier correctly classifies spam emails with probability 0.95 and incorrectly classifies non-spam emails as spam with probability 0.05

The overall probability that an email is spam is 0.3

If an email is classified as spam, what is the probability that it is actually spam?

Start by defining events:

A ~ classifier labels email as spam, S~the email is actually spam

Write probabilities in the problem in terms of events:

$$\mathbb{P}(A|S) = 0.95,\ \mathbb{P}(A|\overline{S}) = 0.05,\ \mathbb{P}(S) = 0.3$$

Write the probability of what we are looking for:

$$\mathbb{P}(S|A) = \frac{\mathbb{P}(A|S)\mathbb{P}(S)}{\mathbb{P}(\boldsymbol{A})} = \frac{\mathbb{P}(A|S)\mathbb{P}(S)}{\mathbb{P}(\boldsymbol{A}|\boldsymbol{S})\mathbb{P}(\boldsymbol{S}) + \mathbb{P}(\boldsymbol{A}|\overline{\boldsymbol{S}})\mathbb{P}(\overline{\boldsymbol{S}})}$$

The classifier correctly classifies spam emails with probability 0.95 and incorrectly classifies non-spam emails as spam with probability 0.05

The overall probability that an email is spam is 0.3

If an email is classified as spam, what is the probability that it is actually spam?

Start by defining events:

A ~ classifier labels email as spam, S~the email is actually spam

Write probabilities in the problem in terms of events:

$\mathbb{P}(A|S) = 0.95$, $\mathbb{P}(A|\overline{S}) = 0.05$, $\mathbb{P}(S) = 0.3$

Write the probability of what we are looking for:

$$\mathbb{P}(S|A) = \frac{\mathbb{P}(A|S)\mathbb{P}(S)}{\mathbb{P}(A)} = \frac{\mathbb{P}(A|S)\mathbb{P}(S)}{\mathbb{P}(A|S)\mathbb{P}(S) + \mathbb{P}(A|\overline{S})\mathbb{P}(\overline{S})} = \frac{0.95 \cdot 0.3}{0.95 \cdot 0.3 + 0.05 \cdot 0.7}$$

More questions/topics to review?