independence + chain rule

LECTURE 6

INDEPENDENCE OF TWO EVENTS (IF PROB. > 0): $\mathbb{P}(A|B) = \mathbb{P}(A) \Leftrightarrow \mathbb{P}(B|A) = \mathbb{P}(B) \Leftrightarrow \mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$
PAIRWISE INDEPENDENCE: Every pair of 2 events is independent
MUTUAL INDEPENDENCE: For every subset of events, P(A n B n …)=P(A)P(B)…
CONDITIONAL INDPENDENCE: A and B conditionally independent on C if $\mathbb{P}(A \cap B|C) = \mathbb{P}(A|C) \cdot \mathbb{P}(B|C)$
CHAIN RULE: $\mathbb{P}(A_1 \cap A_2 \cap \cdots \cap A_n) = \mathbb{P}(A_1) \cdot \mathbb{P}(A_2|A_1) \ldots \mathbb{P}(A_{n-1}|A_1 \cap \cdots \cap A_{n-2}) \cdot \mathbb{P}(A_n|A_1 \cap \cdots \cap A_{n-1})$

INDEPENDENCE

INDEPENDENCE OF 2 EVENTS

"Knowing A doesn't tell us anything about whether B happened"
"Knowing B doesn't tell us anything about whether A happened"

If $\mathbb{P}(A)$ ***and*** $\mathbb{P}(B)$ ***are*** > 0, the events A and B are ***independent*** <u>if</u>

$$\mathbb{P}(A|B) = \mathbb{P}(A) \Leftrightarrow \mathbb{P}(B|A) = \mathbb{P}(B) \Leftrightarrow \mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$$

If one of $\mathbb{P}(A)$ ***or*** $\mathbb{P}(B)$ ***is*** 0, the events A and B are ***independent***
because $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B) = 0$

We flip a fair coin three times. Each flip is independent.
> Is E={HHH} independent of F="at most two heads"?

> Are A="the first flip is heads" and B="the second flip is tails" independent?

ANALYZING INDEPENDENCE OF 3 OR MORE EVENTS

PAIRWISE INDEPENDENCE (CHECK INDEPENDENCE IN EVERY <u>PAIR</u>)

Events $A_1, A_2, \ldots, A_n$ are pairwise independent if
$\mathbb{P}(\boldsymbol{A_i} \cap \boldsymbol{A_j}) = \mathbb{P}(\boldsymbol{A_i}) \cdot \mathbb{P}(\boldsymbol{A_j})$ for all $\boldsymbol{i}, \boldsymbol{j}$

MUTUAL INDEPENDENCE (CHECK INDEPENDENCE IN EVERY <u>SUBSET</u>)

Events $A_1, A_2, \ldots, A_n$ are mutually independent if
$\mathbb{P}(\boldsymbol{A_{i_1}} \cap \boldsymbol{A_{i_2}} \cap \cdots \cap \boldsymbol{A_{i_k}}) = \mathbb{P}(\boldsymbol{A_{i_1}}) \cdot \mathbb{P}(\boldsymbol{A_{i_2}}) \cdots \mathbb{P}(\boldsymbol{A_{i_k}})$
for every subset $\{\boldsymbol{i_1}, \boldsymbol{i_2}, \ldots, \boldsymbol{i_k}\}$ of $\{\boldsymbol{1}, \boldsymbol{2}, \ldots, \boldsymbol{n}\}$.

Roll two fair dice (one red one blue) independently.
R="red die is 3", B="blue die is 5", S="sum is 7"

Are these events *pairwise independent*?
(check every **pair** of events)

Are these events *mutually independent*?
(check every **subset** of events)

Roll a fair 8-sided die.
A = {1,2,3,4}, B = {2,4,6,8}, C = {2,3,5,7}

Are these events *pairwise independent*?
(check every **pair** of events)

Are these events *mutually independent*?
(check every **subset** of events)

Assume alarms from the sensors are *mutually independent*. Probability each sensor triggers is:
P(A)=0.3, P(B)=0.4, P(C)=0.5

What is the probability that all of the sensors go off?

What is the probability that at least one of the sensors go off?

CONDITIONAL INDEPENDENCE

Two events A, B are ***conditionally*** independent on C if
$$\mathbb{P}(A \cap B|C) = \mathbb{P}(A|C) \cdot \mathbb{P}(B|C)$$

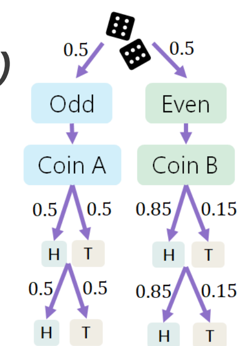
"after we know C, knowing A doesn't give us any additional information about B"

"in this new "restricted sample space" after conditioning on C, A and B are independent"

Coin A is fair, coin B is heads with probability 0.85.
If the die is odd flip A twice (independently); otherwise flip B twice (independently)
C1 ~"the first flip is heads", C2 ~ "the second flip is heads", O ~ "the die was odd"

Are C1 and C2 independent?

Are C1 and C2 *conditionally* independent on O?

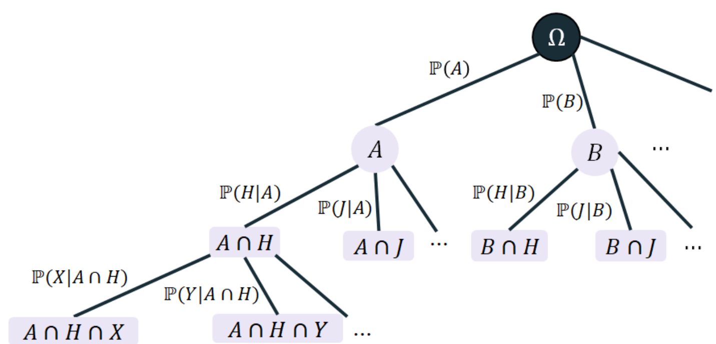


CHAIN RULE

$$\mathbb{P}(A_1 \cap A_2 \cap \cdots \cap A_n)$$
$$= \mathbb{P}(A_1) \cdot \mathbb{P}(A_2|A_1) \ldots \mathbb{P}(A_{n-1}|A_1 \cap \cdots \cap A_{n-2}) \cdot \mathbb{P}(A_n|A_1 \cap \cdots \cap A_{n-1})$$

Let A be the event "The top card is a K♦"
Let B be the event "the second card is a J ♠"
Let C be the event "the third card is a 5 ♠"

What is P(A ∩ B ∩ C)?



Chain Rule: The probability of one of the "nodes" is product of the probabilities of the steps that led to it:
$\mathbb{P}(A \cap H \cap X) = \mathbb{P}(A)\mathbb{P}(H \mid A)\mathbb{P}(X|A \cap H)$

LoTP: Sum up probabilities of intersection with other events partitioning the sample space
$\mathbb{P}(H) = \mathbb{P}(A \cap H) + \mathbb{P}(B \cap H) + \ldots = \mathbb{P}(A)\mathbb{P}(H \mid A) + \mathbb{P}(B)\mathbb{P}(H|B)$