etherpad.wikimedia.org/p/312 for (anonymous) questions/comments!

Independence + Chain Rule

CSE 312 24Su

Lecture 6

Fun application of conditional probability! (Monty Hall Problem)

There are 3 doors (1 door is a prize and the other 2 are goats) and you must pick which door to open. You pick a door randomly. The host opens one of the other two doors to reveal a goat. If you now switch which door you open, your chances of winning increases from 1/3 to 2/3!

Logistics

> **HW2** is out is due on Wednesday (July 3rd)

> **Fill out midterm conflict form by Thursday (July 4th) night**

-- more information / resources for the midterm will be posted end of week

$\mathbb{P}(A|B)$ versus $\mathbb{P}(A \cap B)$

$\mathbb{P}(A \cap B)$ is the probability that both A and B happen

we need to consider the probabilities that both of these events happen and are *not* given any information

"what's the probability A and B happens"

$\mathbb{P}(A|B)$ is the probability that A happens given/conditioning on B

$$\frac{P(A \cap B)}{P(B)}$$

we are looking for the probability that A happens but we are given the extra information that B happens

"what's the probability of A given B"

"if B, what's the probability A happens"

A Technical Note

After you condition on an event, what remains is a probability space.

$(\Omega, \mathbb{P})$ is a valid probability space and, $A \subseteq \Omega$

$(A, \mathbb{P}(\cdot \mid A))$ is valid probability space

Sum of probabilities of outcomes in a sample space is 1:

$\sum_{\omega \in \Omega} \mathbb{P}(\omega) = 1$

$\mathbb{P}(\bar{E}) = 1 - \mathbb{P}(E)$

Sum of probabilities in this conditioned probability space is 1:

$\sum_{\omega \in A} \mathbb{P}(\omega \mid A) = 1$

$\mathbb{P}(\bar{E} \mid A) = 1 - \mathbb{P}(E \mid A)$

⚠️ **Careful!** $\mathbb{P}(E \mid \bar{A}) \neq 1 - \mathbb{P}(E \mid A)$ because this changes the sample space!

A Quick Technical Remark

I often see students write things like

$\mathbb{P}([A|B]|C)$ $\qquad$ $P(A|B|C)$

Thinking something like "probability of A given B given we also know C

This is not a thing.

You probably want $\mathbb{P}(A|[B \cap C])$

"probability of A given both B and C

$A|B$ isn't an event – it's describing an event **and** telling you to restrict the sample space. So you can't ask for the probability of that conditioned on something else.

Summary + Tips

- Today, we talked about **conditional probability**
  - Definition of conditional probability: $\boxed{P(A|B) = \frac{P(A \cap B)}{P(B)}}$
  - Bayes' Theorem: $P(A|B) = \frac{P(B|A)P(A)}{P(B)}$
  - Law of Total Probability: $P(A) = P(A|E_1)P(E_1) + \cdots + P(A|E_n)P(E_n)$ if $E_1, E_2, \ldots E\_n$ partition the sample space
- Now that we're getting into more complex probabilities, it's very helpful to clearly define events and then write given values in terms of the events
  - Writing down all the information given in the problem/what we're asked for in terms of those events is helpful to figure out what rules we can use to relate them together

Outline for *Today*

Last time we talked about **conditional probability** – looking at probabilities where we are given some additional information

Today, we are looking at some more aspects/rules with probabilities
these are all things that conditional probabilities allows us to think about!

> Independence

> Chain Rule

> Conditional Independence

Independence

Definition of Independence

$P(A \mid B) = P(A)$

We've calculated conditional probabilities.

Sometimes conditioning – getting some partial information about the outcome – doesn't actually change the probability.

i.e., knowing B happened doesn't change the probability A happened

We already saw an example like this…

from last Friday's lecture

Conditioning Practice

Red die 6 conditioned on sum 7 1/6

Red die 6 conditioned on sum 9 1/4

Sum 7 conditioned on red die 6 1/6

Red die 6 has probability 1/6 before or after conditioning on sum 7.

	D2=1	D2=2	D2=3	D2=4	D2=5	D2=6
D1=1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1.6)
D1=2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
D1=3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
D1=4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
D1=5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
D1=6	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

Independence

Independence (if $\mathbb{P}(A)$ and $\mathbb{P}(B)$ are > 0)

Two events A, B are *independent* if

$$\mathbb{P}(A|B) = \mathbb{P}(A)$$

Independence

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"Knowing B doesn't tell us anything about whether A happened"

Independence

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(B|A)\mathbb{P}(A)}{\mathbb{P}(B)} = \mathbb{P}(A)$$

Independence **(if $\mathbb{P}(A)$ and $\mathbb{P}(B)$ are > 0)**

Two events A, B are *independent* if

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"Knowing B doesn't tell us anything about whether A happened"

Independence

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(B|A)\mathbb{P}(A)}{\mathbb{P}(B)} = \mathbb{P}(A)$$

$$\mathbb{P}(B|A)\mathbb{P}(A) = \mathbb{P}(A)\mathbb{P}(B)$$

Independence **(if $\mathbb{P}(A)$ and $\mathbb{P}(B)$ are > 0)**

Two events A, B are *independent* if

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Independence

Independence (if $\mathbb{P}(A)$ and $\mathbb{P}(B)$ are > 0)

Two events A, B are *independent* if

$$\mathbb{P}(A|B) = \mathbb{P}(A)$$
$$\Leftrightarrow$$
$$\mathbb{P}(B|A) = \mathbb{P}(B)$$

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(B|A)\mathbb{P}(A)}{\mathbb{P}(B)} = \mathbb{P}(A)$$

$$\mathbb{P}(B|A)\mathbb{P}(A) = \mathbb{P}(A)\mathbb{P}(B)$$

$$\mathbb{P}(B|A) = \mathbb{P}(B)$$

"Knowing B doesn't tell us anything about whether A happened"

Independence

Independence (if $\mathbb{P}(A)$ and $\mathbb{P}(B)$ are > 0)

Two events A, B are *independent* if

$$\mathbb{P}(A|B) = \mathbb{P}(A)$$
$$\Leftrightarrow$$
$$\mathbb{P}(B|A) = \boxed{\mathbb{P}(B)}$$

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(B|A)\mathbb{P}(A)}{\mathbb{P}(B)} = \mathbb{P}(A)$$

$$\mathbb{P}(B|A)\mathbb{P}(A) = \mathbb{P}(A)\mathbb{P}(B)$$

$$\mathbb{P}(B|A) = \mathbb{P}(B)$$

"Knowing B doesn't tell us anything about whether A happened"

"Knowing A doesn't tell us anything about whether B happened"

Independence

Independence (if $\mathbb{P}(A)$ and $\mathbb{P}(B)$ are > 0)

Two events A, B are *independent* if

$$\mathbb{P}(A|B) = \mathbb{P}(A)$$
$$\Leftrightarrow$$
$$\mathbb{P}(B|A) = \mathbb{P}(B)$$

$$\boxed{\mathbb{P}(A|B) = \frac{\mathbb{P}(B|A)\mathbb{P}(A)}{\mathbb{P}(B)} = \mathbb{P}(A)}$$

$$\mathbb{P}(B|A)\mathbb{P}(A) = \mathbb{P}(A)\mathbb{P}(B)$$

$$\boxed{\mathbb{P}(B|A) = \mathbb{P}(B)}$$

$$\frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)} = \mathbb{P}(B)$$

"Knowing B doesn't tell us anything about whether A happened"

"Knowing A doesn't tell us anything about whether B happened"

Independence

Independence (if $\mathbb{P}(A)$ and $\mathbb{P}(B)$ are > 0)

Two events A, B are *independent* if

$$\mathbb{P}(A|B) = \mathbb{P}(A)$$
$$\Leftrightarrow$$
$$\mathbb{P}(B|A) = \mathbb{P}(B)$$
$$\Leftrightarrow$$
$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$$

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(B|A)\mathbb{P}(A)}{\mathbb{P}(B)} = \mathbb{P}(A)$$

$$\mathbb{P}(B|A)\mathbb{P}(A) = \mathbb{P}(A)\mathbb{P}(B)$$

$$\mathbb{P}(B|A) = \mathbb{P}(B)$$

$$\frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)} = \mathbb{P}(B)$$

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$$

"Knowing B doesn't tell us anything about whether A happened"

"Knowing A doesn't tell us anything about whether B happened"

Independence

Independence

If $\mathbb{P}(A)$ *and* $\mathbb{P}(B)$ *are* > 0, the events A and B are *independent* if

$$\mathbb{P}(A|B) = \mathbb{P}(A) \Leftrightarrow \mathbb{P}(B|A) = \mathbb{P}(B) \Leftrightarrow \mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$$

If one of $\mathbb{P}(A)$ *or* $\mathbb{P}(B)$ *is* 0, the events A and B are *independent*

because $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B) = 0$

$P(A) = 0$

You'll sometimes see this called "statistical independence" to emphasize that we're talking about probabilities (not, say, physical interactions).

Examples

We flip a fair coin three times. Each flip is independent.

Is $E = \{HHH\}$ independent of $F =$"at most two heads"?

Are $A =$"the first flip is heads" and $B =$"the second flip is tails" independent?

Examples

We flip a fair coin three times. Each flip is independent.

$2 \cdot 2 \cdot 2$

Is $E = \{HHH\}$ independent of $F =$"at most two heads"?

$\mathbb{P}(E \cap F) \; ? \; \mathbb{P}(E)\mathbb{P}(F)$

$0 \neq \frac{1}{8} \cdot \frac{7}{8}$

NO!

Are $A =$"the first flip is heads" and $B =$"the second flip is tails" independent?

$\mathbb{P}(A \cap B) \; ? \; \mathbb{P}(A)\mathbb{P}(B)$

$\frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2}$

yes!

Examples

Is $E = \{HHH\}$ independent of $F =$"at most two heads"?

$\mathbb{P}(E \cap F) = 0$ (can't have all three heads and at most two heads).

$\mathbb{P}(E) = 1/8$, $\mathbb{P}(F) = 7/8$, $\mathbb{P}(E \cap F) \neq \mathbb{P}(E)\mathbb{P}(F)$. **Not** independent.

Are $A =$"the first flip is heads" and $B =$"the second flip is tails" independent?

$\mathbb{P}(A \cap B) = 2/8$ (uniform measure, and two of eight outcomes meet both A and B.

$\mathbb{P}(A) = 1/2$, $\mathbb{P}(B) = 1/2$, $\frac{2}{8} = \frac{1}{2} \cdot \frac{1}{2}$. These are independent!

Hey Wait

I said "the flips are independent" why aren't E, F independent?

"the flips are independent" means events like <the first flip is blah>" is independent of events like <the second flip is blah>
A ="the first flip is heads" and B ="the second flip is tails" are independent

But if you have an event that involves both flip one and two that might not be independent of an event involving flip one or two.

Intuitively, does knowing whether one event happened tell us something about whether the other event happened?
In this case, if we knew E happened (HHH), we know for sure that F did not happen – there are not 3 > 2 heads

Two Truths and a Lie!

Two of these statements are true, one is false. Explain to each other which ones are true, and find a counter-example to the false one.

1. If A, B both have nonzero probability and are mutually exclusive, then are <u>not</u> independent.

2. If A has zero probability, then A, B are independent (for any B).

3. If two events are independent, then at least one has nonzero probability.

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Two Truths and a Lie!

Two of these statements are true, one is false. Explain to each other which ones are true, and find a counter-example to the false one.

❓ 1. If A, B both have nonzero probability and are **mutually exclusive**, then are not independent.

A happen → B not happen

$P(A \cap B) = 0 \quad P(A)P(B) > 0$

$P(A|B) = 0$

$P(A) > 0$

❓ 2. If A has zero probability, then A, B are independent (for any B).

❓ 3. If two events are independent, then at least one has nonzero probability.

Two Truths and a Lie!

Two of these statements are true, one is false. Explain to each other which ones are true, and find a counter-example to the false one.

✅ 1. If A, B both have nonzero probability and are **mutually exclusive**, then are not independent.

Two mutually exclusive events are never independent. Knowing one event occurred tells us that the other definitely did not also occur.

Mathematically, $\mathbb{P}(A \cap B) = 0$, while $\mathbb{P}(A)\mathbb{P}(\text{B}) > 0$

❓ 2. If A has zero probability, then A, B are independent (for any B).

✓ $P(A \cap B) = 0 \quad P(A)P(B) = 0$

❓ 3. If two events are independent, then at least one has nonzero probability.

Two Truths and a Lie!

Two of these statements are true, one is false. Explain to each other which ones are true, and find a counter-example to the false one.

✅ 1. If A, B both have nonzero probability and are **mutually exclusive**, then are not independent.

Two mutually exclusive events are never independent. Knowing one event occurred tells us that the other definitely did not also occur.

Mathematically, $\mathbb{P}(A \cap B) = 0$, while $\mathbb{P}(A)\mathbb{P}(\mathrm{B}) > 0$

✅ 2. If A has zero probability, then A, B are independent (for any B).

If $\mathbb{P}(A) = 0$, then $\mathbb{P}(A)\mathbb{P}(B) = 0$ and $\mathbb{P}(A \cap B) = 0$. So, $\mathbb{P}(A)\mathbb{P}(B) = \mathbb{P}(A \cap B)$.

❓ 3. If two events are independent, then at least one has nonzero probability.

False $P(A \cap B) = 0$ $P(A)P(B) = 0$

Two Truths and a Lie!

Two of these statements are true, one is false. Explain to each other which ones are true, and find a counter-example to the false one.

✅1. If A, B both have nonzero probability and are **mutually exclusive**, then are <u>not</u> independent.

Two mutually exclusive events are <u>never</u> independent. Knowing one event occurred tells us that the other definitely did not also occur.

Mathematically, $\mathbb{P}(A \cap B) = 0$, while $\mathbb{P}(A)\mathbb{P}(\mathrm{B}) > 0$

✅2. If A has zero probability, then A, B are independent (for any B).

If $\mathbb{P}(A) = 0$, then $\mathbb{P}(A)\mathbb{P}(B) = 0$ and $\mathbb{P}(A \cap B) = 0$. So, $\mathbb{P}(A)\mathbb{P}(B) = \mathbb{P}(A \cap B)$.

❌3. If two events are independent, then at least one has nonzero probability.

If both have probability 0, they are still independent because $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B) = 0$

Independence of 3 or more events

Pairwise independence and **mutual** independence

Independence for 3 or more events

For three or more events, we need two kinds of independence

Pairwise Independence

Events $A_1, A_2, \ldots, A_n$ are pairwise independent if

$$\mathbb{P}(A_i \cap A_j) = \mathbb{P}(A_i) \cdot \mathbb{P}(A_j) \text{ for all } i, j$$

$P(A_i | A_j) = P(A_i)$

Independence for 3 or more events

For three or more events, we need two kinds of independence

Pairwise Independence

Events $A_1, A_2, \ldots, A_n$ are pairwise independent if
$\mathbb{P}(\boldsymbol{A_i} \cap \boldsymbol{A_j}) = \mathbb{P}(\boldsymbol{A_i}) \cdot \mathbb{P}(\boldsymbol{A_j})$ for all $\boldsymbol{i}, \boldsymbol{j}$

Mutual Independence

Events $A_1, A_2, \ldots, A_n$ are mutually independent if
$\mathbb{P}(\boldsymbol{A_{i_1}} \cap \boldsymbol{A_{i_2}} \cap \cdots \cap \boldsymbol{A_{i_k}}) = \mathbb{P}(\boldsymbol{A_{i_1}}) \cdot \mathbb{P}(\boldsymbol{A_{i_2}}) \cdots \mathbb{P}(\boldsymbol{A_{i_k}})$
for every subset $\{\boldsymbol{i_1}, \boldsymbol{i_2}, \ldots, \boldsymbol{i_k}\}$ of $\{\boldsymbol{1}, \boldsymbol{2}, \ldots, \boldsymbol{n}\}$.

Pairwise Independence vs. Mutual Independence

Roll two fair dice (one red one blue) independently

$R =$"red die is 3"

$B =$"blue die is 5"

$S =$"sum is 7"

How should we describe these events?

Pairwise Independence

R ="red die is 3"
B ="blue die is 5"
S ="sum is 7"

R, B, S are pairwise independent

$\mathbb{P}(R \cap B) \mathrel{?}= \mathbb{P}(R)\mathbb{P}(B)$

$\frac{1}{36} = \frac{1}{6} \cdot \frac{1}{6}$ Yes! (These are also independent by the problem statement)

$\mathbb{P}(R \cap S) \mathrel{?}= \mathbb{P}(R)\mathbb{P}(S)$

$\frac{1}{36} \mathrel{?}= \frac{1}{6} \cdot \frac{1}{6}$ Yes!

$\mathbb{P}(B \cap S) \mathrel{?}= \mathbb{P}(B)\mathbb{P}(S)$

$\frac{1}{36} \mathrel{?}= \frac{1}{6} \cdot \frac{1}{6}$ Yes!

Since all three pairs are event independent, we say the ~~random variables~~ are **pairwise independent.**

Mutual Independence

R ="red die is 3"
B ="blue die is 5"
S ="sum is 7"

6

R, B, S are mutually independent?

$\mathbb{P}(R \cap B \cap S) = 0$

$\mathbb{P}(R)\mathbb{P}(B)\mathbb{P}(S) =$

$\frac{1}{6} \cdot \frac{1}{6} \cdot \frac{6}{36}$

Mutual Independence

R ="red die is 3"
B ="blue die is 5"
S ="sum is 7"

R, B, S are not mutually independent.

$\mathbb{P}(R \cap B \cap S) = 0$; if the red die is 3, and blue die is 5 then the sum is 8 (so it can't be 7)

$$\mathbb{P}(R)\mathbb{P}(B)\mathbb{P}(S) = \left(\frac{1}{6}\right)^3 = \frac{1}{216} \neq 0$$

Checking Mutual Independence

It's not enough to check just $\mathbb{P}(\boldsymbol{A} \cap \boldsymbol{B} \cap \boldsymbol{C})$ either. For example...

Roll a fair 8-sided die.

Let A be $\{1,2,3,4\}$

B be $\{2,4,6,8\}$

C be $\{2,3,5,7\}$

$$\mathbb{P}(A \cap B \cap C) = \mathbb{P}(\{2\}) = \frac{1}{8}$$

$$\mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$$

Checking Mutual Independence

It's not enough to check just $\mathbb{P}(A \cap B \cap C)$ either.

Roll a fair 8-sided die.

Let A be $\{1,2,3,4\}$

B be $\{2,4,6,8\}$

C be $\{2,3,5,7\}$

$$\mathbb{P}(A \cap B \cap C) = \mathbb{P}(\{2\}) = \frac{1}{8}$$

$$\mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$$

$$P(B \cap C) \neq \underbrace{P(B)}_{\frac{4}{8}}\underbrace{P(C)}_{\frac{4}{8}}$$

$$\frac{3}{8} \neq \frac{4}{8} \cdot \frac{4}{8}$$

But B and C aren't independent. **Because there's a <u>subset</u> that's not independent,** A, B, C are <u>not</u> mutually independent.

Checking Pairwise / Mutual Independence

To check pairwise independence of events:

Check **every** subset of size two.

To check mutual independence of events:

Check **every** subset.

If a set of events are mutually independent, it is also pairwise independent.

but not necessarily the other way around!

How do these properties help us?

Sensors (A, B, and C) in a factory monitor temperature, pressure, and humidity. Let events A, B, and C be the events that sensors A, B, and C start an alarm respectively. Assume alarms from the sensors are mutually independent. The probability that each sensor triggers an alarm is:

$$\mathbb{P}(A) = 0.3,\ \ \mathbb{P}(B) = 0.4,\ \ \mathbb{P}(C) = 0.5$$

What is the probability that **all** of the sensors go off?

$$\mathbb{P}(A \cap B \cap C) = P(A)P(B)P(C) = 0.3 \cdot 0.4 \cdot 0.5$$

What is the probability that **at least one** of the sensors go off?

$$\mathbb{P}(A \cup B \cup C) = 1 - \underbrace{P(\bar{A} \cap \bar{B} \cap \bar{C})}_{\text{NONE GO OFF}} = 1 - \underbrace{P(\bar{A})}_{0.7} \cdot \underbrace{P(\bar{B})}_{0.6} \cdot \underbrace{P(\bar{C})}_{0.5}$$

How do these properties help us?

Sensors (A, B, and C) in a factory monitor temperature, pressure, and humidity. Let events A, B, and C be the events that sensors A, B, and C start an alarm respectively. Assume alarms from the sensors are <u>mutually independent</u>. The probability that each sensor triggers an alarm is:

$$\mathbb{P}(A) = 0.3,\ \ \mathbb{P}(B) = 0.4,\ \ \mathbb{P}(C) = 0.5$$

What is the probability that **all** of the sensors go off?

$$\mathbb{P}(A \cap B \cap C) \underset{\text{mutual independence}}{=} \mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C) = 0.3 \cdot 0.4 \cdot 0.5 = 0.06$$

What is the probability that **at least one** of the sensors go off?

How do these properties help us?

Sensors (A, B, and C) in a factory monitor temperature, pressure, and humidity. Let events A, B, and C be the events that sensors A, B, and C start an alarm respectively. Assume alarms from the sensors are <u>mutually independent</u>. The probability that each sensor triggers an alarm is:

$$\mathbb{P}(A) = 0.3,\ \mathbb{P}(B) = 0.4,\ \mathbb{P}(C) = 0.5$$

What is the probability that **all** of the sensors go off?

$$\mathbb{P}(A \cap B \cap C) \xrightarrow{\text{mutual independence}} \mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C) = 0.3 \cdot 0.4 \cdot 0.5 = 0.06$$

What is the probability that **at least one** of the sensors go off?

$$\mathbb{P}(A \cup B \cup C) = \mathbb{P}\left((A^C \cap B^C \cap C^C)^C\right) = 1 - \mathbb{P}(A^C \cap B^C \cap C^C) = 1 - (0.7 \cdot 0.6 \cdot 0.5) = 0.79$$

"probability at least one sensor goes off is 1 – probability <u>none</u> of the sensors go off"

Conditional Independence

Conditional Independence $P(A|B \cap C) = P(A|C)$

$P(A \cap B) = P(A)P(B)$

Conditional Independence

Two events A, B are *conditionally* independent on C if

$$\mathbb{P}(A \cap B|C) = \mathbb{P}(A|C) \cdot \mathbb{P}(B|C)$$

> "if you condition on C, they are independent"

> "in this new "restricted sample space" after conditioning on C, A and B are independent"

> "after we know C happened knowing A doesn't give us any additional information about whether B happens"

Example

You have two coins. ~~Coin A is~~ fair, coin B comes up heads with probability 0.85.

You will roll a (fair) die, if the result is odd flip coin A twice (independently); if the result is even flip coin B twice (independently)

Let C_1 be the event "the first flip is heads", C_2 be the event "the second flip is heads", O be the event "the die was odd"

Are C_1 and C_2 independent?

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(Unconditioned) Independence

check whether $\mathbb{P}(\boldsymbol{C_1} \cap \boldsymbol{C_2}) = \mathbb{P}(\boldsymbol{C_1})\mathbb{P}(\boldsymbol{C_2})$

$\mathbb{P}(C_1) =$

Coin A is fair, coin B is heads with probability 0.85.
If the die is odd flip coin A twice (independently); otherwise flip coin B twice (independently)

C_1 ~*"the first flip is heads"*
C_2 ~ *"the second flip is heads"*
O ~ *"the die was odd"*

(Unconditioned) Independence

check whether $\mathbb{P}(\boldsymbol{C_1} \cap \boldsymbol{C_2}) = \mathbb{P}(\boldsymbol{C_1})\mathbb{P}(\boldsymbol{C_2})$

$$\mathbb{P}(C_1) = \underbrace{\mathbb{P}(C_1|O)\mathbb{P}(O)}_{\text{fair}} + \underbrace{\mathbb{P}(C_1|\bar{O})\mathbb{P}(\bar{O})}_{\text{biased}} = \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot 0.85 = .675$$

Coin A is fair, coin B is heads with probability 0.85.
If the die is odd flip coin A twice (independently); otherwise flip coin B twice (independently)

C_1 ~*"the first flip is heads"*
C_2 ~ *"the second flip is heads"*
O ~ *"the die was odd"*

(Unconditioned) Independence

check whether $\mathbb{P}(\boldsymbol{C_1} \cap \boldsymbol{C_2}) = \mathbb{P}(\boldsymbol{C_1})\mathbb{P}(\boldsymbol{C_2})$

$$\mathbb{P}(C_1) = \mathbb{P}(C_1|O)\mathbb{P}(O) + \mathbb{P}(C_1|\bar{O})\mathbb{P}(\bar{O}) = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot 0.85 = .675$$

$\mathbb{P}(C_2) = .675$ (the same formula works)

Coin A is fair, coin B is heads with probability 0.85.
If the die is odd flip coin A *twice (independently); otherwise flip coin* B *twice (independently)*

C_1 ~*"the first flip is heads"*
C_2 ~ *"the second flip is heads"*
O ~ *"the die was odd"*

(Unconditioned) Independence

check whether $\underline{\mathbb{P}(\boldsymbol{C_1} \cap \boldsymbol{C_2})} = \underbrace{\mathbb{P}(\boldsymbol{C_1})\mathbb{P}(\boldsymbol{C_2})}$

$$\mathbb{P}(C_1) = \mathbb{P}(C_1|O)\mathbb{P}(O) + \mathbb{P}(C_1|\bar{O})\mathbb{P}(\bar{O}) = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot 0.85 = .675$$

$\mathbb{P}(C_2) = .675$ (the same formula works)

$$\mathbb{P}(C_1)\mathbb{P}(C_2) = \boxed{.675^2} = \underline{.455625}$$

Coin A is fair, coin B is heads with probability 0.85.
If the die is odd flip coin A twice (independently); otherwise flip coin B twice (independently)

C_1 ~*"the first flip is heads"*
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(Unconditioned) Independence

check whether $\mathbb{P}(\boldsymbol{C_1} \cap \boldsymbol{C_2}) = \mathbb{P}(\boldsymbol{C_1})\mathbb{P}(\boldsymbol{C_2})$

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$\mathbb{P}(C_1 \cap C_2) \neq \mathbb{P}(C_1)\mathbb{P}(C_2)$ So, they're not independent!

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***Intuition**: seeing a head gives you information – information that it's more likely you got the biased coin and so the next head is more likely.*

Conditional Independence on O

check whether $\mathbb{P}(C_1 \cap C_2 | O) = \mathbb{P}(C_1|O)\mathbb{P}(C_2|O)$

$\mathbb{P}(C_1|O) = 1/2$

$\mathbb{P}(C_2|O) = 1/2$

$\mathbb{P}(C_1|O)\mathbb{P}(C_2|O) = \frac{1}{2} \cdot \frac{1}{2} = 1/4$

$\mathbb{P}(C_1 \cap C_2|O) = \frac{1}{4}$

if the die is odd, there are 2 independent flips from A

$\mathbb{P}(C_1|O)\mathbb{P}(C_2|O) = \mathbb{P}(C_1 \cap C_2|O)$

Coin A is fair, coin B is heads with probability 0.85.
If the die is odd flip coin A twice (independently); otherwise flip coin B twice (independently)

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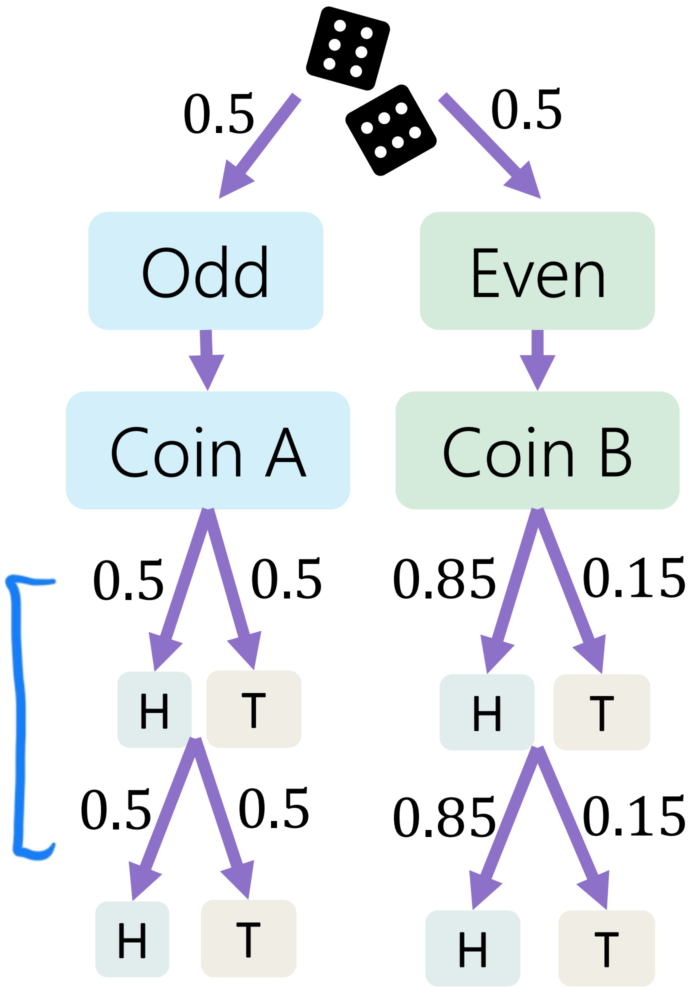
Yes! C_1 and C_2 are conditionally independent, conditioned on O.

once we know which coin is being flipped, there are 2 independent flips

Takeaway

from the example we just did



Read the problem carefully!

> When we say "these steps are independent of each other" in a sequential process, we usually mean "*conditioned on all prior steps*, these steps are conditionally independent of each other."

> Without this conditioning, the steps are often dependent because they can provide information about the chosen path.

Chain Rule

How to compute $\mathbb{P}(A_1 \cap A_2 \cap \cdots \cap A_n)$ if there aren't mutually independent?

Chain Rule

We defined conditional probability as: $\boxed{\mathbb{P}(A_1|A_2) = \frac{\mathbb{P}(A_1 \cap A_2)}{\mathbb{P}(A_2)}}$

Which means $\mathbb{P}(A_1 \cap A_2) = \mathbb{P}(A_1|A_2)\mathbb{P}(A_2)$

$$\text{So, } \mathbb{P}((A_1 \cap A_2) \cap A_3) = \mathbb{P}(A_1 \cap A_2)\mathbb{P}(A_3|A_1 \cap A_2)$$

$$= \mathbb{P}(A_1)\mathbb{P}(A_2|A_1)\mathbb{P}(A_3|A_1 \cap A_2)$$

extending this for n events….

Chain Rule

We defined conditional probability as: $\mathbb{P}(A|B) = \frac{\mathbb{P}(A\cap B)}{\mathbb{P}(B)}$

Which means $\mathbb{P}(A \cap B) = \mathbb{P}(A|B)\mathbb{P}(B)$

Chain Rule

$\mathbb{P}(A_1 \cap A_2 \cap \cdots \cap A_n)$ $P(A_3|A_1 \cap A_2)$ ...

$= \mathbb{P}(A_1) \cdot \mathbb{P}(A_2|A_1) \ldots \mathbb{P}(A_{n-1}|A_1 \cap \cdots \cap A_{n-2}) \cdot \mathbb{P}(A_n|A_1 \cap \cdots \cap A_{n-1})$

We have n steps/tasks, and want the probability they all occur
> multiply the probability of each step, conditioning on all previous steps

Chain Rule Example

Shuffle a standard deck of 52 cards (so every ordering is equally likely).

Let A be the event "The top card is a K♦"

Let B be the event "the second card is a J ♠

Let C be the event "the third card is a 5 ♠

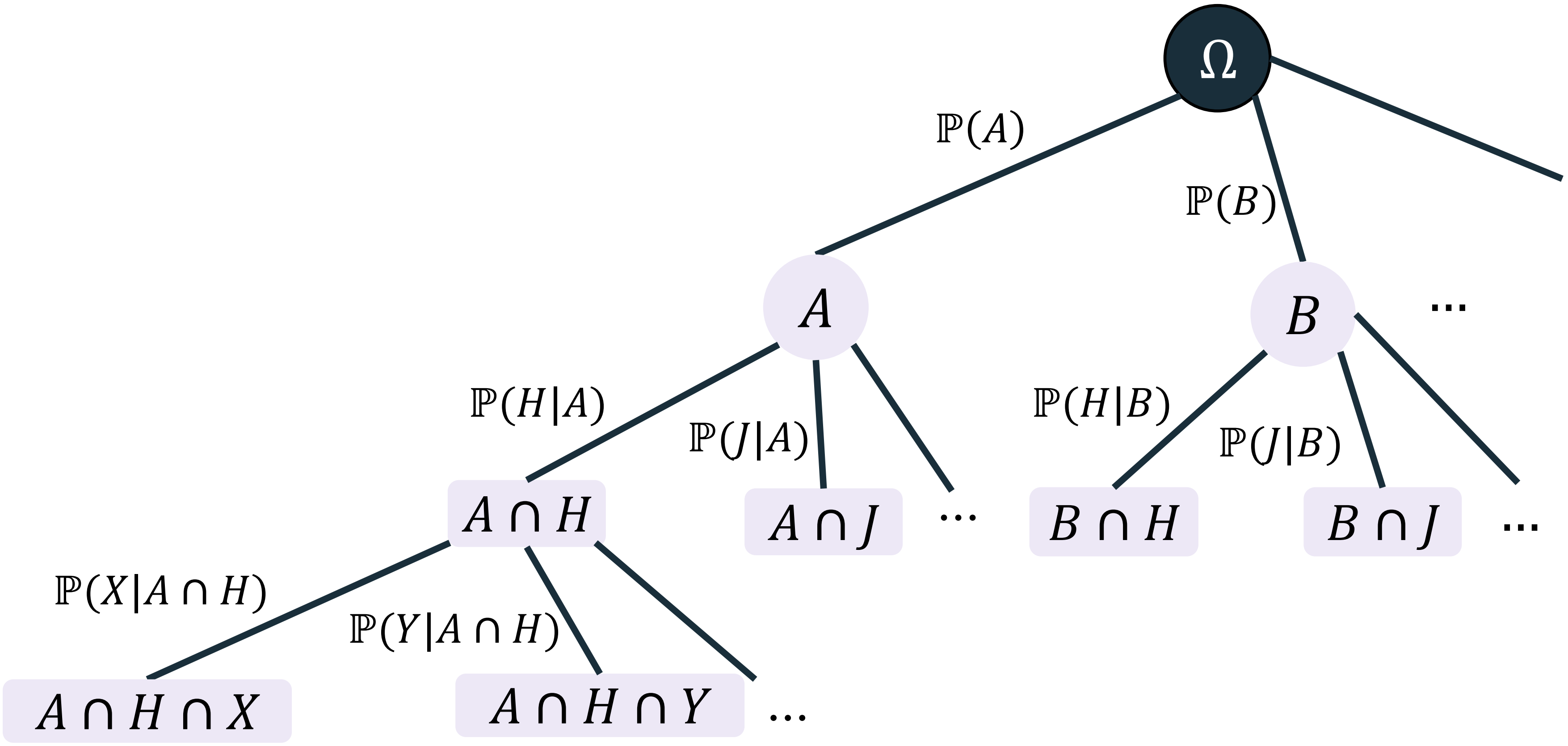
What is $\mathbb{P}(A \cap B \cap C)$?

Use the chain rule!

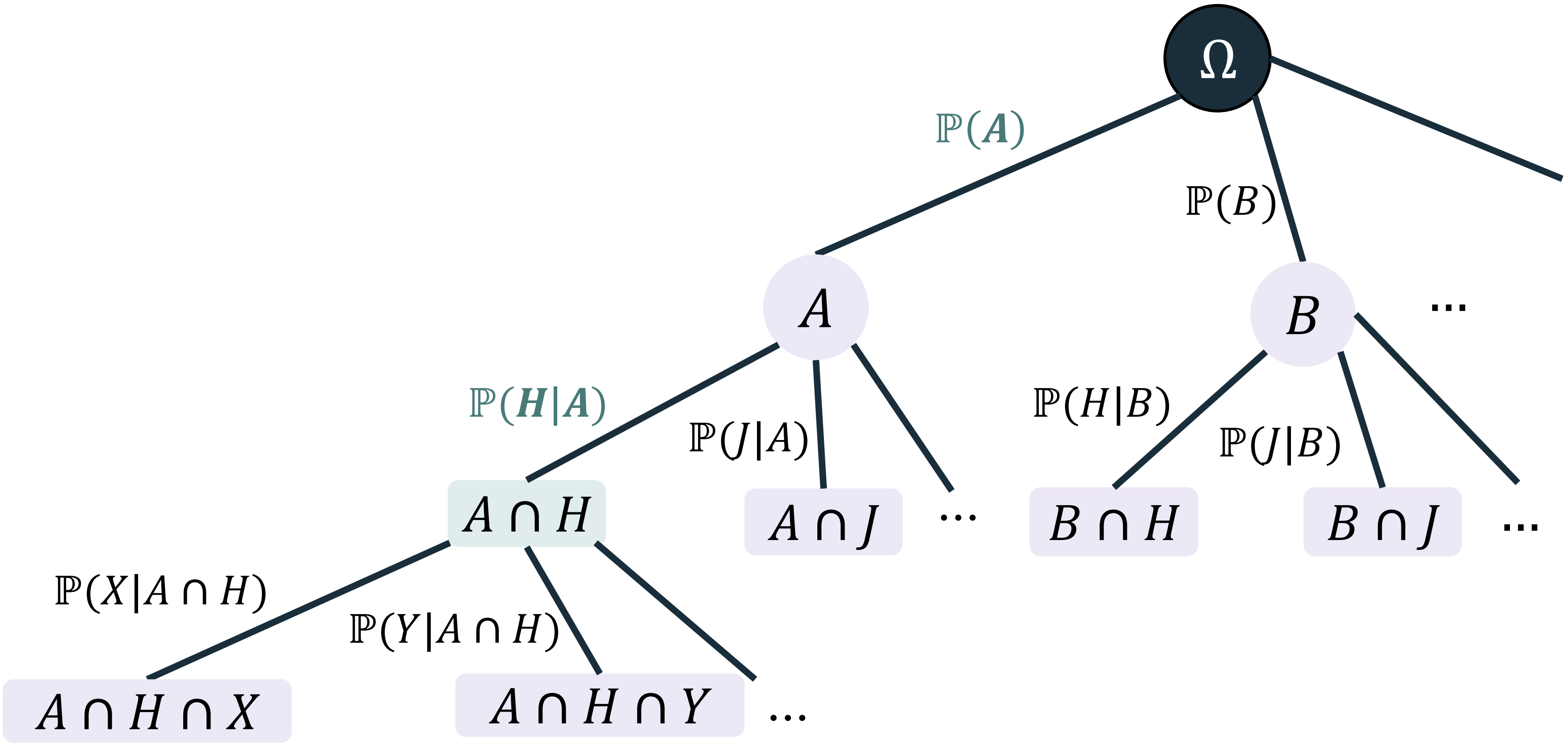
$$\mathbb{P}(A) \cdot \mathbb{P}(B|A) \cdot \mathbb{P}(C|A \cap B)$$

$$\frac{1}{52} \cdot \frac{1}{51} \cdot \frac{1}{50}$$

A Visual for the Chain Rule (and LoTP!)



A Visual for the Chain Rule (and LoTP!)



Chain Rule: The probability of one of the "nodes" (e.g.,$\mathbb{P}(A \cap H)$) is the product of the probabilities of each of the steps that led to it:

$$\mathbb{P}(A \cap H) = \mathbb{P}(A)\mathbb{P}(H|A)$$

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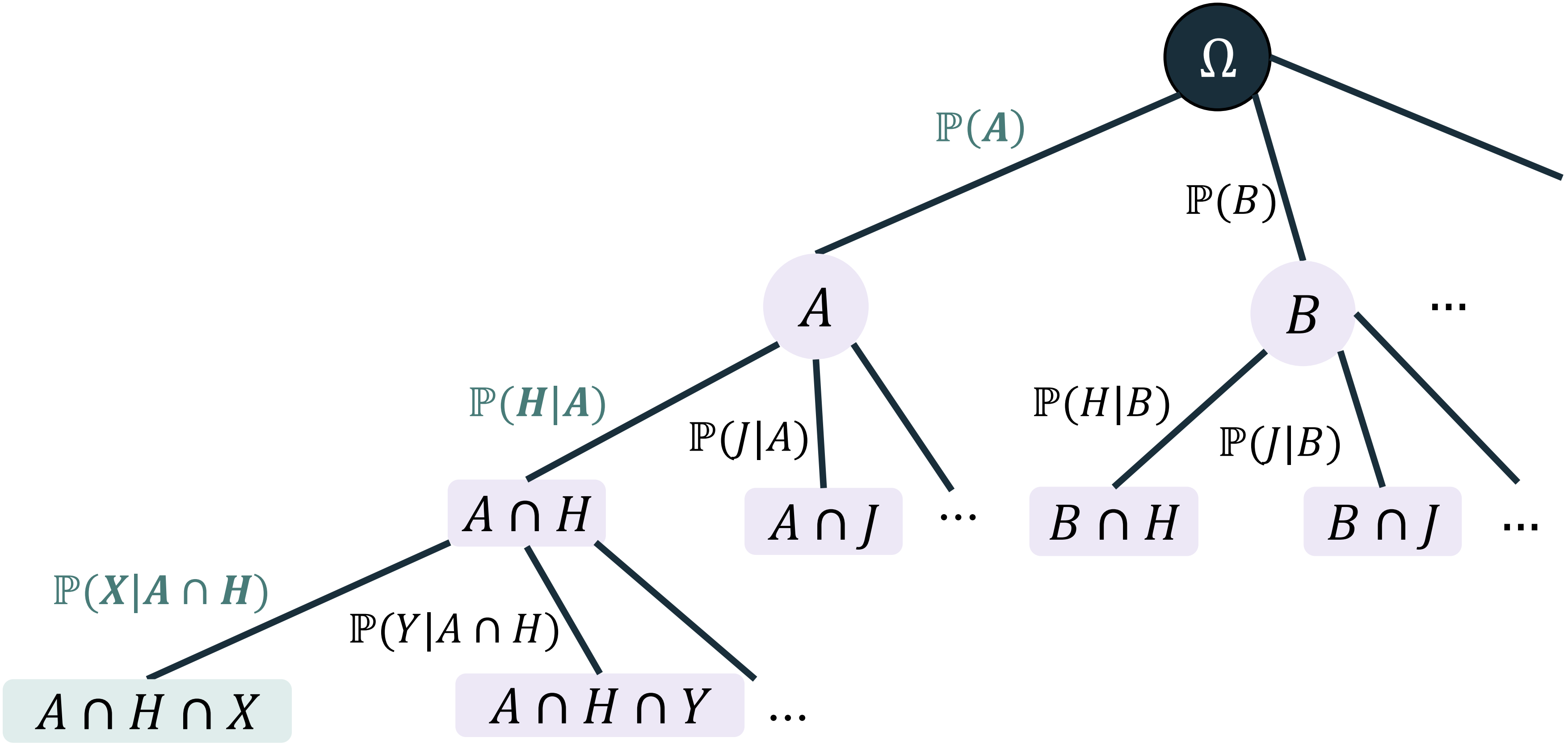
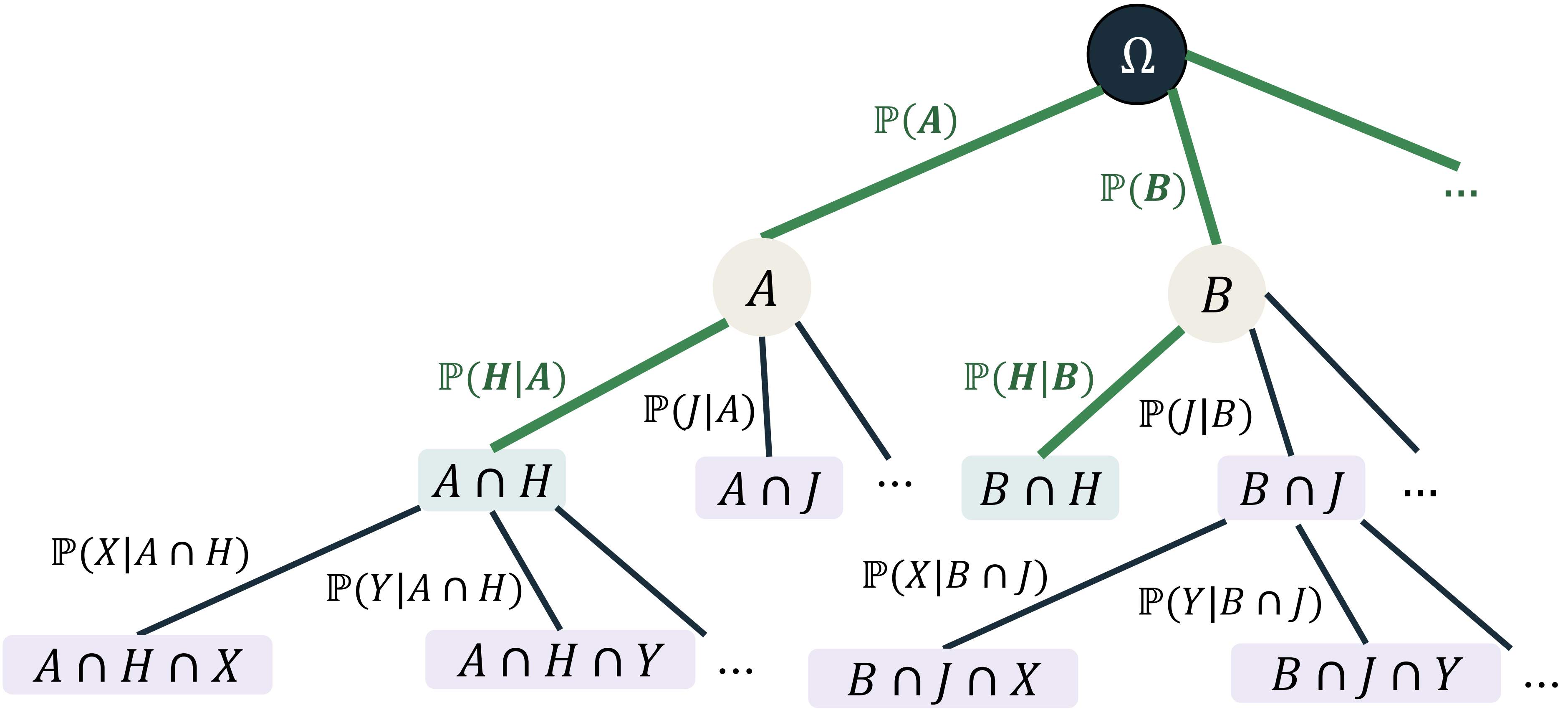


Chain Rule: The probability of one of the "nodes" (e.g.,$\mathbb{P}(A \cap H \cap X)$) is the product of the probabilities of each of the steps that led to it:

$$\mathbb{P}(A \cap H \cap X) = \mathbb{P}(A \cap H)\mathbb{P}(X|A \cap H) = \mathbb{P}(A)\mathbb{P}(H|A)\mathbb{P}(X|A \cap H)$$

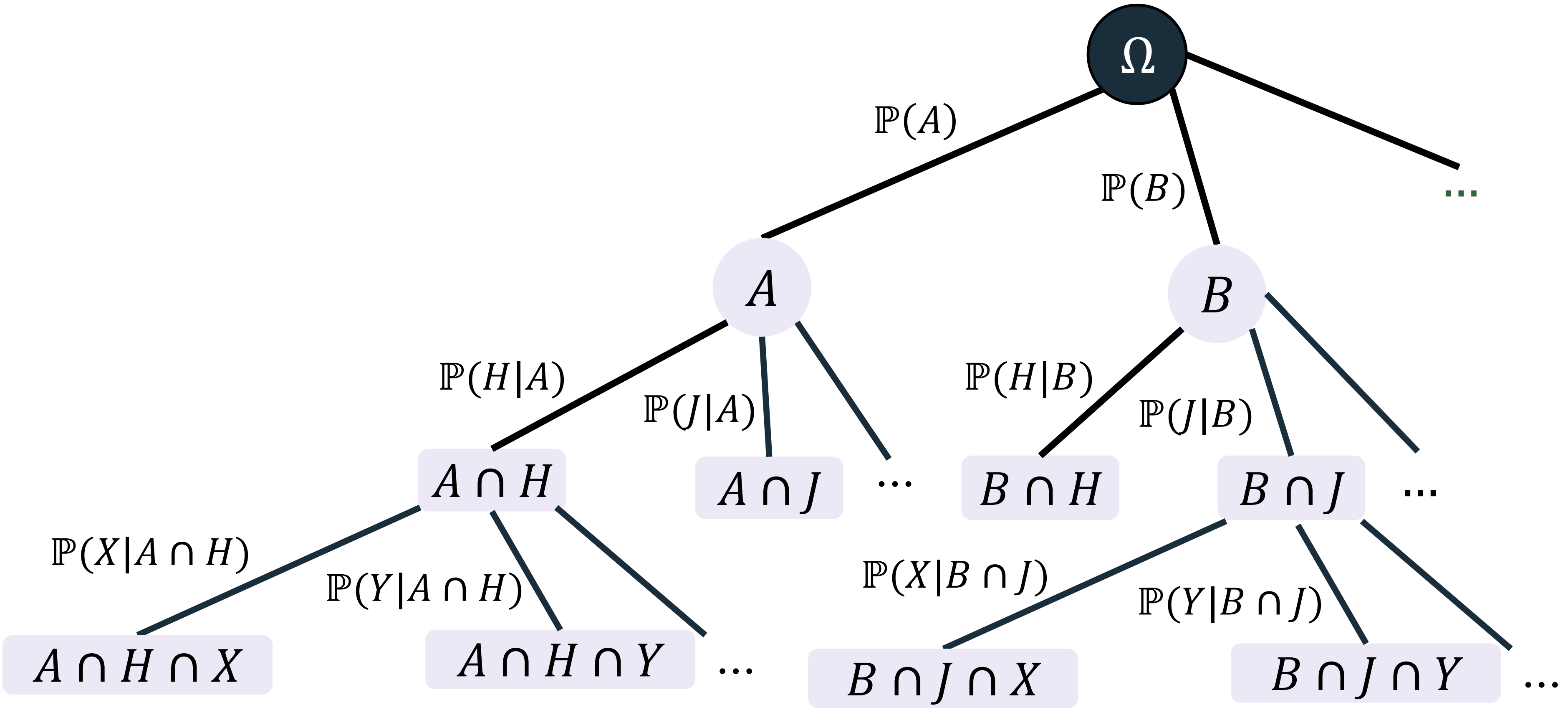
A Visual for the Chain Rule (and LoTP!)



Law of Total Probability: Sum up probabilities of intersection with other events

$$\mathbb{P}(H) = \mathbb{P}(A \cap H) + \mathbb{P}(B \cap H) + \cdots = \mathbb{P}(A)\mathbb{P}(H|A) + \mathbb{P}(B)\mathbb{P}(H|B)$$

A Visual for the Chain Rule (and LoTP!)



⚠️ on homeworks, you can use these kinds of visuals to help with reasoning, but make sure to still explicitly show applications of chain rule, LoTP, etc.) ⚠️

Summary

Today, we talked about

> Independence

independence between 2 events, pairwise and mutual independence for >2 events, events being *conditionally independent* on a third event

> Chain Rule

probability of an intersection of events/set of tasks or steps that aren't mutually independent

On Wednesday, we are doing a <u>breather/catch-up lecture</u>! We will:

> Review what has been covered so far

> Talk about how to parse long problems, mentally process all the pieces

> Answer questions/talk about what you all want to!

So, bring questions! Questions don't have to be specific, even just asking to go over a piece of content again is perfectly valid.

Setting the stage: Random Variables

Implicitly defining Ω

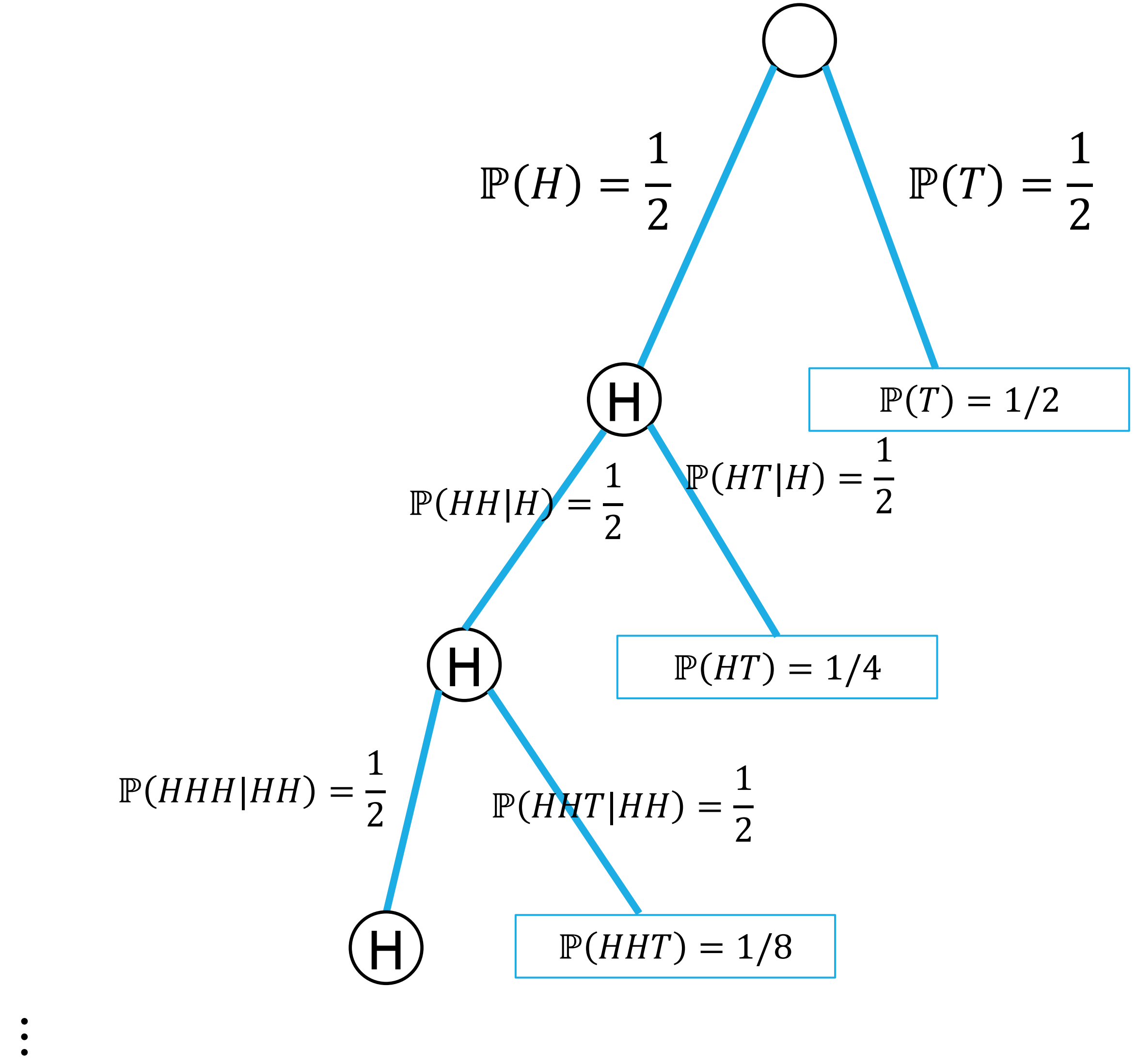
We've often skipped an explicit definition of Ω.

Often $|\Omega|$ is infinite, so we really couldn't write it out (even in principle).

How would that happen?

Flip a fair coin (independently each time) until you see your first tails.
what is the probability that you see at least 3 heads?

An infinite process.



Ω is infinite.

A sequential process is also going to be infinite…

But the tree is "self-similar"

From every node, the children look identical (H with probability ½, continue pattern; T to a leaf with probability ½)

Finding $\mathbb{P}$(at least 3 heads)

Method 1: infinite sum.

Ω includes H^iT for every i. Every such outcome has probability $1/2^{i+1}$

What outcomes are in our event?

$\sum_{i=3}^{\infty} 1/2^{i+1} = \frac{\frac{1}{2^4}}{1-1/2} = \frac{1}{8}$

Infinite geometric series, where common ratio is between -1 and 1 has closed form $\frac{\text{first term}}{1-\text{ratio}}$

Finding $\mathbb{P}$(at least 3 heads)

Method 2:

Calculate the complement

$\mathbb{P}$(at most 2 heads) = $\frac{1}{2} + \frac{1}{4} + \frac{1}{8}$

$\mathbb{P}$(at least 3 heads)= $1 - \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8}\right) = \frac{1}{8}$