Probability

CSE 312 23Su

Lecture 4

Announcements

- HW1 is due tonight (11:59pm)
- HW2 will come out this evening, due next Wednesday.

Office Hours schedule is on the calendar on the webpage

1-1s with me/TAs (see information on Ed)

Cards

A lot of counting problems deal with cards!

A "standard" deck of cards has 52 cards $(13 \cdot 4 = 52)$.

Each card has one of **4 suits**

diamonds ♦,

hearts ♥,

clubs ♧,

spades ♠

and one of **13 values/ranks** (Ace,2,3,4,5,6,7,8,9,10,Jack,Queen, King).

e.g., **Ace♦, 5♦, 5♧, 10♠** are all possible cards

A "k-card-hand" is an unordered set of k cards

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A "k-card-hand" is an unordered set of k cards

How many five-card "flushes" are there? – a flush is a hand of cards all of the same suit. (e.g., {A♠, 3♠, 5♠, 6♠, Q♠})

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Five-card "flushes"

How many five-card "flushes" are there? – a flush is a hand of cards all of the same suit.

Think: How would I create a set of cards that is a flush?

Way 1:

1. Pick the suit (e.g., ♠) – $\binom{\mathbf{4}}{\mathbf{1}}$

2. Pick the specific values/cards from that suit (e.g., {A,3,5,6,Q}) - $\binom{\mathbf{13}}{\mathbf{5}}$

Now we've created an unordered 5-card flush! (e.g., {A♠, 3♠, 5♠, 6♠, Q♠})

$$\binom{4}{1} \cdot \binom{13}{5}$$

Five-card "flushes"

Way 2:

Pretend order matters.

1. Pick any first card – 52 options
2. All remaining cards must be from the same suit of that first suit:

   12 options for the 2nd card, 11 options for the 3rd card, etc.

Divide out the overcounting - divide by $5!$, since order isn't supposed to matter (i.e., only count each unordered flush once)

$$\frac{52 \cdot 12 \cdot 11 \cdot 10 \cdot 9}{5!}$$

This equals the same number as what we got on the last slide!

How many 5-card hands have <u>at least</u> 3 aces?

There are 4 Aces (and 48 non aces) in a deck of cards

1. **Choose 3 aces**: $\binom{4}{3}$

2. Then **pick 2 of the 49 remaining cards** to form a 5(the last ace is allowed as well, because we're allowed to have all 4): $\binom{49}{2}$

$\binom{4}{3} \cdot \binom{49}{2}$

What's wrong with this calculation? Does it,

A) Overcount B) Undercount C) It's correct! D) I have no idea :)

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Sleuth’s Criterion
How to check if we counted correctly?

For each outcome that we want to count, there should be exactly one set of choices in the sequential process that will lead to that outcome.

> If there are no sequence of choices that will lead to the outcome, we have **undercounted**.

> If there is more than one sequence of choices that will lead to the outcome, we have **overcounted**.

Sleuth’s Criterion (in context)

How to check if we counted correctly?

For each “5-card hands with at least 3 aces” that we want to count, there should be exactly one set of choices in the sequential process that will lead to that outcome.

> If there are no sequence of choices that will lead to a particular 5-card hand with at least 3 aces, we have **undercounted**.

> If there is more than one sequence of choices that will lead to a particular **5-card hand with at least 3 aces**, we have **overcounted**.

How many 5-card hands have at least 3 aces?

For each "5-card hands with at least 3 aces" that we want to count, there should be exactly one set of choices in the sequential process that will lead to that outcome.

A♧, A♠, A♦, Q♥, K ♠ is a valid outcome should counted exactly once.

Step 1 (choose 3 aces): {A♧, A♠, A♦}
Step 2 (pick 2 of remaining 49): {Q♥, K ♠}

How many 5-card hands have at least 3 aces?

For each "5-card hands with at least 3 aces" that we want to count, there should be exactly one set of choices in the sequential process that will lead to that outcome.

A♧, A♠, A♦, Q♥, K ♠ is a valid outcome should counted exactly once.

Step 1 (choose 3 aces): {A♧, A♠, A♦}
Step 2 (pick 2 of remaining 49): {Q♥, K ♠}

Great! There's no other set of choices that will lead to this hand.

How many 5-card hands have at least 3 aces?

For each "**5-card hands with at least 3 aces**" that we want to count, there should be exactly one set of choices in the sequential process that will lead to that outcome.

A♧, A♠, A♦, A♥, K ♠ is a valid outcome should counted exactly once.

Step 1 (choose 3 aces): {A♧, A♠, A♦}
Step 2 (pick 2 of remaining 49): {A♥, K ♠}

How many 5-card hands have at least 3 aces?

For each "5-card hands with at least 3 aces" that we want to count, there should be exactly one set of choices in the sequential process that will lead to that outcome.

A♧, A♠, A♦, A♥, K ♠ is a valid outcome should counted exactly once. But…

Step 1 (choose 3 aces): {A♧, A♠, A♦}
Step 2 (pick 2 of remaining 49): {A♥, K ♠}

Step 1 (choose 3 aces): {A♧, A♥, A♦}
Step 2 (pick 2 of remaining 49): {A ♠, K ♠}

Both of these are different choices in the sequential process and are counted separately, but they are the same hand!
This is overcounting ☹

Fixing The Overcounting

How many 5-card hands have at least 3 aces?

Way 1: We could start with our incorrect solution & **subtract the overcounting.**

Our original incorrect solution:
1. **Choose 3 aces**: $\binom{4}{3}$, 2. Then **pick 2 of the 49 remaining cards:** $\binom{49}{2}$ → $\binom{4}{3} \cdot \binom{49}{2}$

What kinds of hands do we overcount (counted many times in the sequential process)?

> 5-card hands with 4 Aces (i.e., a hand like {A♧, A♠, A♦, A♥, X})

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{A♧, A♠, A♦}, {A♥, **X**}
{A♧, A♠, A♥}, {A♦, **X**}
{A♧, A♥, A♦}, {A♠, **X**}
{A♥, A♠, A♦}, {A♧, **X**}

So, how many outcomes are overcounted?

> There are $\binom{4}{4} \cdot 48 = 48$ 5-card hands with all 4 Aces

> Each of these hands is counted 4 times, but we only want to count it once

> So we've counted $(4-1) \cdot 48 = 3 \cdot 48$ processes that shouldn't count.

That would give a corrected total of $\binom{4}{3} \cdot \binom{49}{2} - 3 \cdot 48$

How many 5-card hands have at least 3 aces?

Way 1: We could **subtract out the overcounting** - count exactly which hands are overcounted in our sequential process, and how many times each of those hands are overcounted, and subtract that from our initial count.

$$\binom{4}{3} \cdot \binom{49}{2} - 3 \cdot 48$$

Way 2: Try a different approach! The problem with our original solutions was trying to account for the "at least" - **come up with disjoint sets and count separately.**

Case 1: There are exactly 3 aces: $\binom{4}{3} \cdot \binom{48}{2}$

Case 2: There are exactly (all) 4 aces: $\binom{4}{4} \cdot \binom{48}{1}$

Applying the sum rule: $\binom{4}{3} \cdot \binom{48}{2} + \binom{4}{4} \cdot \binom{48}{1}$

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Does this overcount/undercount?

For a valid outcome, there should be exactly 1 set of choices leading to that outcome:

A♧, A♠, A♦, Q ♧, K ♠ - this will fall under the first case. The only possible set of choices leading to this is {A♧, A♠, A♦} in the 1st step and {Q ♧, K ♠} in the 2nd

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Does this overcount/undercount?

For a valid outcome, there should be exactly 1 set of choices leading to that outcome:

A♣, A♠, A♦, A♥, K ♠ - this will fall under the second case. The only possible set of choices leading to this is {A♣, A♠, A♦, A♥} in the 1st step and {K ♠} in the 2nd

Takeaways

- There are often many ways to do the same problem! When you can do a problem two **very** different ways and get the same answer, you get much more confident in the answer.

- To check for overcounting, try thinkin about some actual outcomes that we want to be counted exactly once and make sure it can be constructed with exactly one set of choices in the sequential process

Today

So far…we've done a lot of counting.

Starting today, we get to calculate **probabilities**!

And the counting techniques we've learned are going to come in handy when computing probabilities ☺

Mostly notation and vocabulary today.

Probability

What is **Probability**?

There are lots of things we aren’t certain about!

Is it going to rain this weekend?

Am I going to get a 6 when rolling this dice?

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Probability is a way of quantifying our uncertainty when more than one outcome is possible

- What is the probability that is rains tomorrow?
- What is the probability that I get a 6 when rolling a dice?

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What is the probability that is rains tomorrow?

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To have "real-world" examples, we're going to make some assumptions:

We can flip a coin, and each face is equally likely to come up

We can roll a die, and every number is equally likely to come up

We can shuffle a deck of cards so that every ordering is equally likely.

Experiment

Experiment

An action or process that leads to one or more outcomes.

A *random* experiment is an experiment where the outcome can't be predicted with certainty beforehand

Examples:

Tossing a fair coin

Rolling a dice

Drawing a name from a hat

Sample Space

Sample Space

A sample space Ω is the set of all possible outcomes of an experiment.

Examples:

For a single coin flip, $\Omega = \{H, T\}$

For a series of two coin flips, $\Omega = \{HH, HT, TH, TT\}$

For rolling a (normal) die: $\Omega = \{1,2,3,4,5,6\}$

Event

Event

An event $E \subseteq \Omega$ is a subset of possible outcomes (i.e. a subset of the sample space Ω)

Examples:

Get a head in one coin flip ($E = \{H\}$)

Get at least one head among two coin flips ($E = \{HH, HT, TH\}$)

Get an even number on a die-roll ($E = \{2,4,6\}$).

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Get at least one head among two coin flips ($E = \{HH, HT, TH\}$)

Get an even number on a die-roll ($E = \{2,4,6\}$).

Notation note: Since sets are usually represented with a single capital letter, we also denote events with a single capital letter (it doesn't have to be E, it can be anything!)

Examples

Experiment:
I roll a blue 4-sided die and a red 4-sided die.

The table contains the sample space.
$\Omega = \{(1,1), (1,2), \ldots\}$

	D2=1	D2=2	D2=3	D2=4
D1=1	(1,1)	(1,2)	(1,3)	(1,4)
D1=2	(2,1)	(2,2)	(2,3)	(2,4)
D1=3	(3,1)	(3,2)	(3,3)	(3,4)
D1=4	(4,1)	(4,2)	(4,3)	(4,4)

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Let A be the event **that "the sum of the dice is even**". The outcomes in this event are in gold

$A = \{(1,1), (1,3), (2,2), (2,4), (3,1), (3,3), (4,2), (4,4)\}$

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$A = \{(1,1), (1,3), (2,2), (2,4), (3,1), (3,3), (4,2), (4,4)\}$

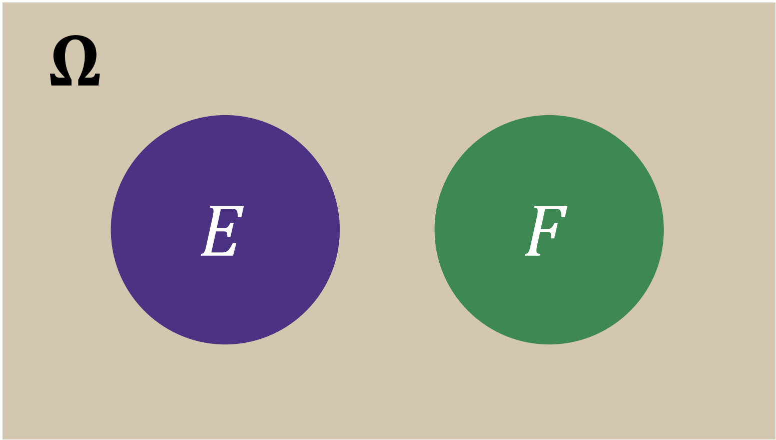
Let B be the event that **"first die is a 1"**. The outcomes in this event are in green.

$B = \{(1,1), (1,2), (1,3), (1,4)\}$

Mutually Exclusive Events

Two events E, F are **mutually exclusive if they can't happen simultaneously**.

In notation, $E \cap F = \emptyset$ (i.e. they're disjoint subsets of the sample space).



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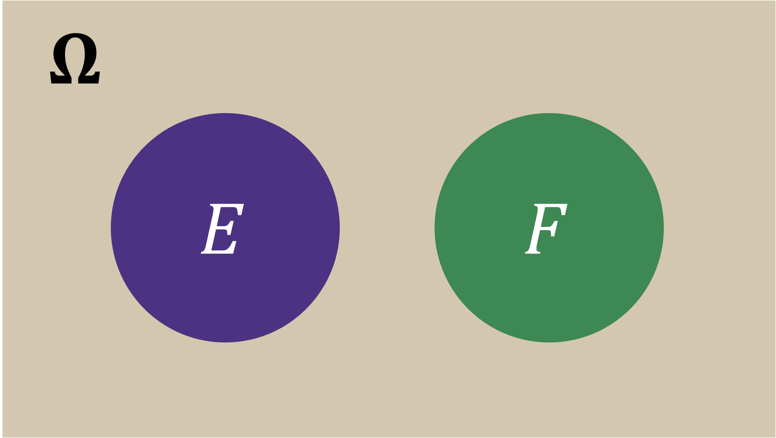
For example, if we flip a coin **and** roll a dice: $\Omega = \{H, T\} \times \{1,2,3,4,5,6\}$

$E_1 =$"the coin came up heads"

$E_2 =$"the coin came up tails"

$E_3 =$"the die showed an even number"



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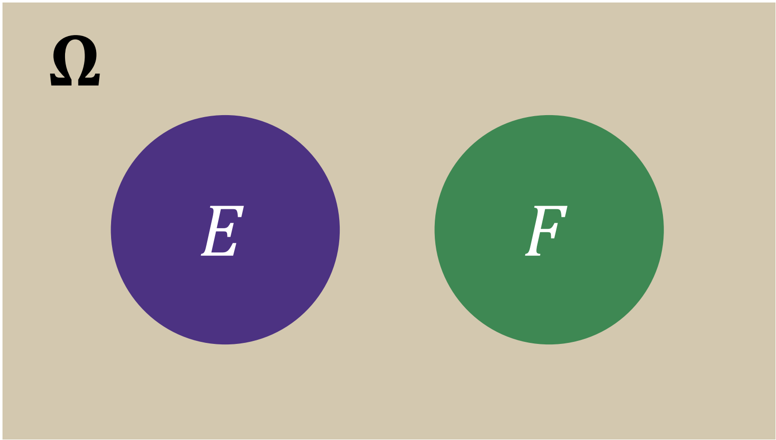
E_1 = "the coin came up heads"

E_2 = "the coin came up tails"

E_3 = "the die showed an even number"

E_1 and E_2 are mutually exclusive.
E_1 and E_3 are not mutually exclusive.

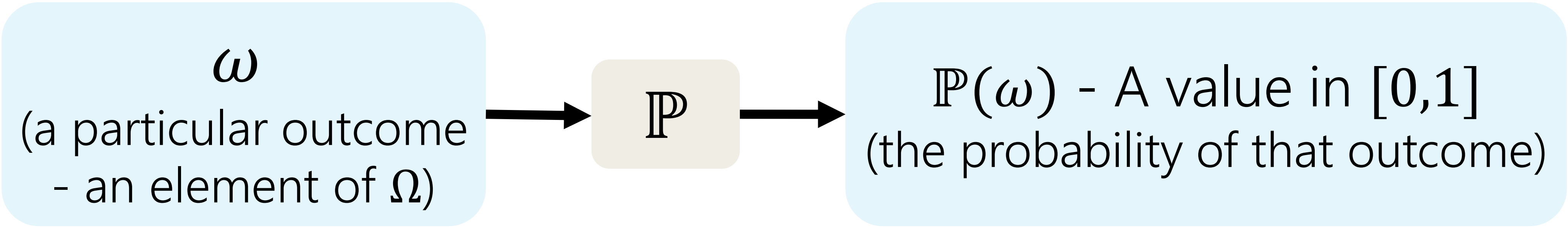


Probability

Probability

A probability is a number between 0 and 1 describing how likely a particular outcome or event is.

Formally, we define a function $\mathbb{P}$ that assigns a probability to every outcome ω in the sample space.



Notation: $\mathbb{P}(\omega)$, $P(\omega)$, $\Pr(\omega)$ are all equivalent!

Example

Imagine we toss one coin.

Our sample space $\Omega = \{H, T\}$

What do you want $\mathbb{P}$ to be?

Recall: $\mathbb{P}$ assigns a probability to each outcome in the sample space

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What do you want $\mathbb{P}$ to be?

Recall: $\mathbb{P}$ assigns a probability to each outcome in the sample space

It depends on what we want to model!

If we have a *fair coin* $\mathbb{P}(H) = \mathbb{P}(T) = \frac{1}{2}$.

But we also might have a *biased coin*: $\mathbb{P}(H) = .85$, $\mathbb{P}(T) = 0.15$.

Probability Space

Probability Space

A **(discrete) probability space** is a pair $(\Omega, \mathbb{P})$ where:

$\boldsymbol{\Omega}$ is the sample space

$\mathbb{P}: \Omega \to [0,1]$ is the probability measure.

$\mathbb{P}$ satisfies:

- $\mathbb{P}(x) \geq 0$ for all x
- $\sum_{\omega \in \Omega} \mathbb{P}(\omega) = 1$

Probability Space

Experiment: Flip a fair coin and roll a fair (6-sided) die.

$\Omega = \{H, T\} \times \{1,2,3,4,5,6\} = \{(H, 1), (H, 2) \dots., (T, 1), (T, 2), \dots\}$

$\mathbb{P}(\omega) = \frac{1}{12}$ for every $\omega \in \Omega$

Is $(\Omega, \mathbb{P})$ a valid probability space?

☑ $\mathbb{P}$ takes in elements of Ω and outputs numbers between 0 and 1

☑ $\sum_{\omega \in \Omega} \mathbb{P}(\omega) = 12 \cdot \frac{1}{12} = 1.$

Probability of An Event?

Formally, the $\mathbb{P}$ takes in only single outcomes. But…we will use the same notation to define the probability of an event (set of outcomes)!

$$\mathbb{P}(E) = \sum_{\omega \in \mathrm{E}} \mathbb{P}(\omega)$$

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Example:

Flip a fair coin and roll a fair (6-sided) die.

$\Omega = \{H, T\} \times \{1,2,3,4,5,6\}$, $\mathbb{P}(\omega) = \frac{1}{12}$ for every $\omega \in \Omega$

Let E be the event the dice is a 2. $E = \{(H, 2), (T, 2)\}$

$\mathbb{P}(E) = \mathbb{P}\big((H, 2)\big) + \mathbb{P}\big((H, 2)\big) = \frac{1}{12} + \frac{1}{12} = \frac{1}{6}$

Probability Facts

Axioms and Consequences

We wrote down 2 requirements (axioms) on probability measures

- $\mathbb{P}(x) \geq 0$ for all x (non-negativity)
- $\sum_{x \in \Omega} \mathbb{P}(x) = 1$ (normalization)

These lead quickly to these **three corollaries**:

- $\mathbb{P}(\bar{E}) = 1 - \mathbb{P}(E)$ (complementation)
- If $E \subseteq F$, then $\mathbb{P}(E) \leq \mathbb{P}(F)$ (monotonicity)
- $\mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F) - \mathbb{P}(E \cap F)$ (inclusion-exclusion)
  - if E and F are mutually exclusive: $\mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F)$

Uniform Probability Spaces

A common type of probability space!

Uniform Probability Space

In a **uniform** probability space $(\Omega, \mathbb{P})$, every outcome in the sample space is **equally likely to occur**. For every outcome $\omega \in \Omega$,

$$\mathbb{P}(\omega) = \ ?$$

Uniform Probability Space

In a **uniform** probability space $(\Omega, \mathbb{P})$, every outcome in the sample space is **equally likely to occur**. For every outcome $\omega \in \Omega$,

$$\mathbb{P}(\omega) = \frac{1}{|\Omega|}$$

For example,

> Flipping a fair coin: for every $\omega \in \Omega$, $\mathbb{P}(\omega) = \frac{1}{2}$

$\Omega = \{1,2,3,4,5,6\}$

> Rolling a fair dice: for every $\omega \in \Omega$, $\mathbb{P}(\omega) = \frac{1}{6}$

$\Omega = \{1,2,3,4,5,6\}$

Uniform Probability Space

In a **uniform** probability space $(\Omega, \mathbb{P})$, every outcome in the sample space is **equally likely to occur**. For every outcome $\omega \in \Omega$,

$$\mathbb{P}(\omega) = \frac{1}{|\Omega|}$$

Finding the probability of an event in a uniform probability space:

$$\mathbb{P}(E) = \sum_{\omega \in E} \mathbb{P}(\omega) = \sum_{\omega \in E} \frac{1}{|\Omega|} = \frac{|E|}{|\Omega|}$$

Uniform Probability Space (summarized ☺)

Uniform Probability Space

A **uniform probability space** is a probability space where all outcomes in the sample space are **equally likely** to occur.

> For every *outcome* $\omega \in \Omega$, $\mathbb{P}(\omega) = \frac{1}{|\Omega|}$

> For an *event* $E \subseteq \Omega$, $\mathbb{P}(E) = \frac{|E|}{|\Omega|}$

Uniform Probability Space

Let your sample space be all possible outcomes of a sequence of 100 coin tosses. Assign the uniform measure to this sample space. What is the probability of the event "there are exactly 50 heads?

A. $\binom{100}{50}/2^{100}$

B. $1/101$

C. $1/2$

D. $1/2^{50}$

E. There is not enough information in this problem.

Fill out the poll everywhere so Claris knows how much to explain
Go to **pollev.com/cse312** and login with your UW identity

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$|\Omega| = 2^{100}$ because each of the 100 coin tosses have 2 options if it is head or tails

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Our probability measure is $\mathbb{P}(\omega) = \frac{1}{2^{100}}$ for every $\omega \in \Omega$

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Let H be the event that there are exactly 50 heads

$|H|= \binom{100}{50}$ because we pick which of the 50 coin tosses are heads – the rest are tails

$$\boldsymbol{P(H) = \frac{|H|}{|\Omega|} = \frac{\binom{100}{50}}{2^{100}}}$$

Uniform Probability Space

Let your sample space be all possible outcomes of a sequence of 100 coin tosses. Assign the uniform measure to this sample space. What is the probability of the event “there are exactly 50 heads?

⚠️ We need to be careful how we define the sample space! ⚠️

For example, if we defined the sample space as the number of heads in the sequence: $\Omega = \{1,2,\ldots,99,100\}$, we can’t use a uniform probability space because every outcome is not equally likely here.

If we want to use a uniform probability, pick a sample space where every outcome is equally likely!

More examples

Mainly focusing on uniform probability spaces

More Examples!

Suppose you roll two dice. Each die is fair and they don't affect each other. What is the probability of both dice being even?

What is the **sample space**?

What is the **probability measure** $\mathbb{P}$?

What is the **event**?

What is the **probability**?

More Examples!

Suppose you roll two dice. Each die is fair and they don't affect each other. What is the probability of both dice being even?

What is the **sample space**? $\Omega = \{1,2,3,4,5,6\} \times \{1,2,3,4,5,6\}$

What is the **probability measure** $\mathbb{P}$? $\mathbb{P}(\omega) = 1/36$ for all $\omega \in \Omega$

What is the **event**? $\{2,4,6\} \times \{2,4,6\}$

What is the **probability**? $3^2/6^2$

More Examples!

Suppose you roll two dice. Each die is fair and they don't affect each other. What is the probability of both dice being even?

What if we defined our sample space as the *unordered* pairs of the die?

What is the **sample space**?

$\{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),(2,2),(2,3),(2,4),(2,5),(2,6)$
$(3,3),(3,4),(3,5),(3,6),(4,4),(4,5),(4,6),(5,5),(5,6),(6,6)\}$

What is the **probability measure** $\mathbb{P}$?

$\mathbb{P}((x,y)) = 2/36$ if $x \neq y$, $\mathbb{P}(x,x) = 1/36$

What is the **event**? $\{(2,2),(4,4),(6,6),(2,4),(2,6),(4,6)\}$

What is the **probability**? $3 \cdot \frac{1}{36} + 3 \cdot \frac{2}{36} = \frac{9}{36}$

Takeaways

There is often more than one sample space possible! But one is probably easier than the others.

Finding a sample space that will make the uniform measure correct will usually make finding the probabilities easier to calculate.

- This often involves deciding what kind of information we need to encode in the sample space (e.g., should we care about order or not?)

Another Example

Suppose you shuffle a deck of cards so any arrangement is equally likely. What is the probability that the top two cards have the same value?

Sample Space:

Probability Measure:

Event:

Probability:

Another Example

Suppose you shuffle a deck of cards so any arrangement is equally likely. What is the probability that the top two cards have the same value?

Sample Space: $\Omega = \{(x, y): x \text{ and } y \text{ are different cards } \}$

Probability Measure: uniform measure $\mathbb{P}(\omega) = \frac{1}{52 \cdot 51}$

Event: all pairs with equal values

Probability: $\frac{13 \cdot P(4,2)}{52 \cdot 51}$

Another Example

Suppose you shuffle a deck of cards so any arrangement is equally likely. What is the probability that the top two cards have the same value?

Sample Space: Set of all orderings of all 52 cards

Probability Measure: uniform measure $\mathbb{P}(\omega) = \frac{1}{52!}$

Event: all lists that start with two cards of the same value

Probability: $\frac{13 \cdot P(4,2) \cdot 50!}{52!}$

Another Example

Suppose you shuffle a deck of cards so any arrangement is equally likely. What is the probability that the top two cards have the same value?

Sample Space: Set of all orderings of all 52 cards

Probability Measure: uniform measure $\mathbb{P}(\omega) = \frac{1}{52!}$

Event: all lists that start with two cards of the same value

Probability: $\frac{13 \cdot P(4,2) \cdot 50 * 49 * 48 * \cdots . * 2 * 1}{52 * 51 * 50 * 49 * 48 * \ldots * 2 * 1}$

Takeaway

There’s often information you “don’t need” in your sample space.

It won’t give you the wrong answer.

But it sometimes makes for extra work/a harder counting problem,

Good indication: you cancelled A LOT of stuff that was common in the numerator and denominator.

Few notes about events and samples spaces

- If you're dealing with a situation where you may be able to use a uniform probability space, make sure to set up the sample space in a way that every outcome is equally likely.

- Try not overcomplicate the sample space – only include the information that you need in it.

- When you define an event, make sure it is a <u>subset</u> of the sample space!

  e.g., if order matters in the sample space, it should also matter in the event space

Some Quick Observations

For discrete probability spaces (the kind we've seen so far)

$\mathbb{P}(E) = 0$ if and only if ?

$\mathbb{P}(E) = 1$ if and only if ?

Some Quick Observations

For discrete probability spaces (the kind we've seen so far)

$\mathbb{P}(E) = 0$ if and only if an event can't happen.

$\mathbb{P}(E) = 1$ if and only if an event is guaranteed (every outcome outside E has probability 0).

Birthday Paradox

Sharing Birthdays 🎂

There are about 50 people in this class. What is the probability that at least 2 of us share a birthday? Assume that there are exactly 365 possible birthdays, and each possibility is equally likely.

We know from the pigeonhole principle that if there are >365 people in the group, there will certainly be at least 2 people that share the same birthday. But what's the *probability* of this happening if there are only 50 people?

Sharing Birthdays

There are about 50 people in this class. What is the probability that at least 2 of us share a birthday? Assume that there are exactly 365 possible birthdays, and each possibility is equally likely.

What do you think this probability is closest to?

A) 0.001

B) 0.5

C) 0.99

D) 1

Fill out the poll everywhere **pollev.com/cse312** and login with your UW identity

Sharing Birthdays 🎂

There are about 50 people in this class. What is the probability that at least 2 of us share a birthday? Assume that there are exactly 365 possible birthdays, and each possibility is equally likely.

Sample Space:

Probability Measure:

Event:

Probability:

There are about 50 people in this class. What is the probability that at least 2 of us share a birthday? Assume that there are exactly 365 possible birthdays, and each possibility is equally likely.

Sample Space: Set of assignments of birthdays to people. $|\Omega| = 365^{50}$

Probability Measure: Uniform probability measure. $\mathbb{P}(\omega) = \frac{1}{365^{50}}$ for $\omega \in \Omega$

Event: Let E be the event that at least 2 people share a birthday.

Probability: $\mathbb{P}(E) = 1 - \mathbb{P}(\bar{E})$. $\bar{E}$ is the event that **no one** shares a birthday.

$$\mathbb{P}(\bar{E}) = \frac{|\bar{E}|}{|\Omega|} = \frac{P(365,50)}{365^{50}}.$$

We use a permutation for $|\bar{E}|$ because birthdays are "selected" without replacement, (all have different birthdays) and order matters (my birthday is different from your birthday, etc.)

$$\mathbb{P}(E) = 1 - \frac{P(365,50)}{365^{50}} \approx 097.$$

Sharing Birthdays 🎂

There are about 50 people in this class. What is the probability that at least 2 of us share a birthday? Assume that there are exactly 365 possible birthdays, and each possibility is equally likely.

$$\mathbb{P}(E) = 1 - \frac{P(365,50)}{365^{50}} \approx 097.$$

This is pretty high! So almost definitely, two of us here share the same birthday 🥳

That’s very likely! Why?

It turns out that human brains find thinking about probabilities difficult!

Our brains are a bit selfish! When it comes to the probability that someone shares our birthday, that would be $\frac{1}{365}$ - not quite so likely.

But if we’re looking at any pair’s birthday in a group of n people, there are $\binom{n}{2}$ pairs of people, which grows quadratically with n. So the probability of at least one pair of people sharing a birthday approaches 1 pretty fast!

Summary

- Probability allows us to assign a value between 0 and 1 to outcomes
- A ***random experiment*** is any process where the outcome is not known for certain
- The ***sample space*** of an experiment is the set of all possible outcomes
- An ***event*** is a subset of the sample space (some set of outcomes)
- The ***probability space*** is the pair $(\Omega, \mathbb{P})$ where Ω is the sample space and $\mathbb{P}$ is the probability measure (a *function* that assigns probabilities to every outcome ω in the sample space)
  - A ***uniform probability space*** is a common type of probability space where every outcome is equally likely. To find the probability of an event in a uniform probability space, we find the size of the event divided by the size of the sample space

Extra Examples

Rolling Dice

Suppose I had a two, fair, 6-sided dice that we roll, one green, one red. What is the probability that we see at least one 3 in the two rolls?

Sample Space:

Probability Measure:

Event:

Probability:

Rolling Dice

Suppose I had a two, fair, 6-sided dice that we roll, one green, one red. What is the probability that we see at least one 3 in the two rolls?

Sample Space: $\{1,2,3,4,5,6\} \times \{1,2,3,4,5,6\}$

$|\Omega|$ is $6 \cdot 6 = 36$ because each of the dice rolls have 6 options.

Probability Measure: $\mathbb{P}(\omega) = \frac{1}{6^2} = \frac{1}{36}$

Event: Let A be the event that we see at least one 3 in the two rolls

Probability: $\mathbb{P}(A) = 1 - \mathbb{P}(\bar{A})$. $\bar{A}$ is the event that neither of the two rolls is a 3.$|\bar{A}| = 5^2 = 25$ because each roll has 5 options. $\mathbb{P}(\bar{A}) = \frac{|\bar{A}|}{|\Omega|} = \frac{25}{36}$.

So, $\mathbb{P}(A) = 1 - \frac{25}{36} = \frac{15}{36}$

Balls and Urns

You have an urn* with two red balls and two green balls inside.

Take out two of the balls replacing the first ball after you take it out.

What's the probability of drawing out both red balls?

Sequential process: $\frac{1}{2}$ probability of the first being red

$\frac{1}{2}$ probability of the second being red.

*An urn is a vase