etherpad.wikimedia.org/p/312 for (anonymous) questions/comments!

Even *More* Counting

CSE 312 23Su

Lecture 3

Fun Fact

The number of possible unique chess games is estimated to be around 10^{120}, which is more than the number of atoms in the universe! To get an idea of how this number gets so big – the white pieces have 20 options for the first move, and the black pieces has 20 options for the second move. After only this first play, there are $20 \cdot 20 = 400$ possibilities. Imagine how this increases after more moves!

Announcements

> HW1 due on Wednesday

Q5 – the dishes should not be repeated

Q6b – small typo in the point in the hint

> Remember to do concept checks!

> Updated OH times posted later today/tomorrow morning

This is a fast-paced class!

There's a <u>lot</u> we cover in this class and the summer quarter is short.

We're here to help <3

- Office hours
- Post on the Edstem board
- Concept checks are helpful – try to do it right after each class
- Ask questions!!
- 1-1s with course staff (coming soon) + don't hesitate to email me

Outline

So Far

Sum and Product Rules

Combinations and Permutations

Introduce ordering and remove it to make calculations easier

This Time

One final counting rule: **Stars and Bars**

Counting proofs!

> **Combinatorial proofs** *(counting by two ways)*

> **Pigeonhole Principle**

Broader takeaways

> How to tell what counting rule to apply

> How do I know we're not over/undercounting?

Stars and Bars

The last counting rule in a while! :')

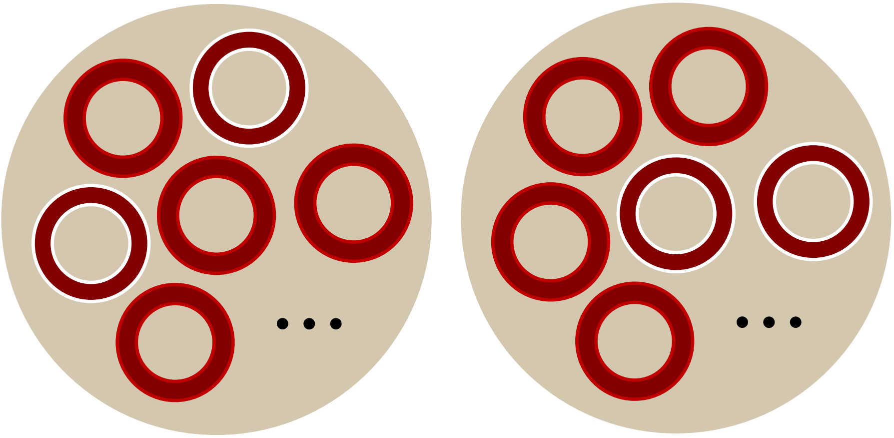
Donuts

You're going to buy one-dozen donuts (i.e., 12 donuts) from chocolate, strawberry, coconut, blueberry, and lemon (i.e. 5 types)

How many different donut boxes can you buy?

- There are unlimited donuts of each type
- Donuts of the same type are indistinguishable
- Order of the donuts in the box does not matter

SC, ... CS ..



Set of all chocolate donuts in the store

What does it mean for the "donuts of the same type to be indistinguishable?

If our dozen donuts includes 2 chocolate donuts, both outcomes on the left should be counted the same

(i.e., because items of the same type are indistinguishable, there's only 1 way to pick k donuts of a particular type)

Donuts – an **incorrect** approach

$\frac{5^{12}}{12!}$ 125

2C,105

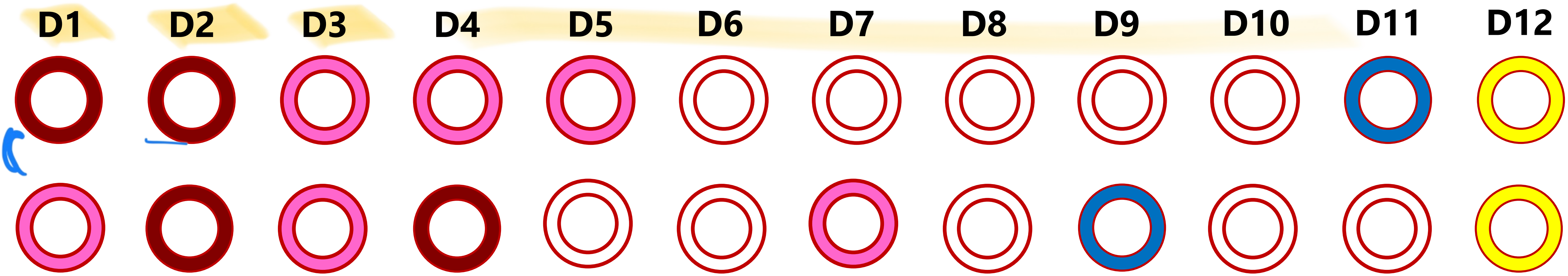
$\binom{12}{2}$

You're going to buy one-dozen donuts (i.e., 12 donuts)

There's chocolate, strawberry, coconut, blueberry, and lemon (i.e. 5 types)

How many different donut boxes can you buy? Consider two boxes the same if they contain the same number of every kind of donut (o.

Idea: 5^{12} *because each of the 12 donuts have 5 options for its flavor?*



Both the above outcomes are equivalent and should only be counted once!

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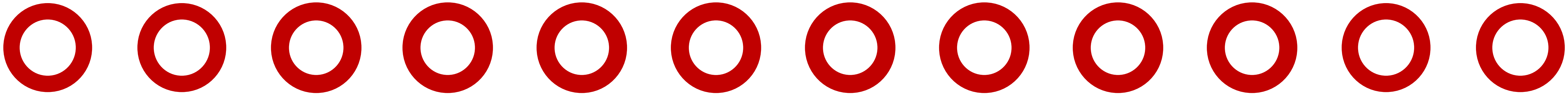
All we care about is how many donuts of each type are ordered



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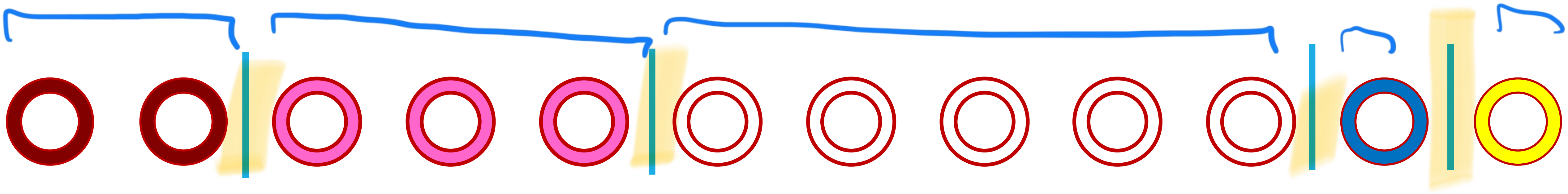

2 chocolate, 3 strawberry, 5 coconut, 1 blueberry, 1 lemon

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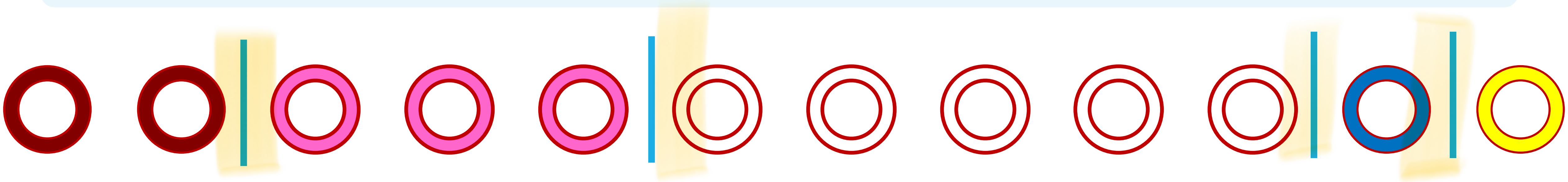
How many different donut boxes can you buy?

> There are unlimited donuts of each type

> Donuts of the same type are indistinguishable

> Order of the donuts in the box does not matter

We only need $\mathbf{5-1=4}$ **dividers to divide** this set of 12 donuts **into** $\mathbf{5}$ **groups**



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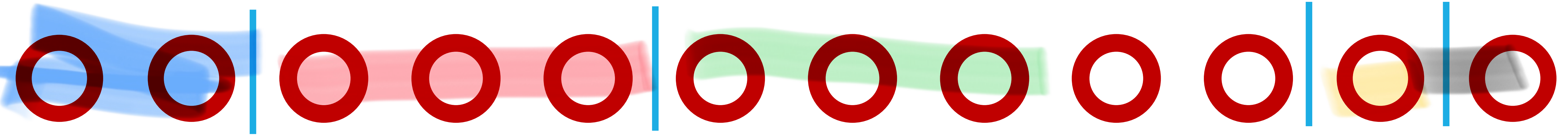

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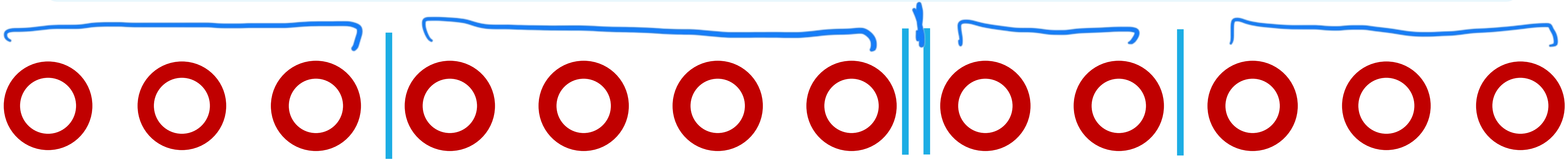

3 chocolate, 4 strawberry, 0 coconut, 2 blueberry, 3 lemon

Donuts

You're going to buy one-dozen donuts (i.e., 12 donuts) from chocolate, strawberry, coconut, blueberry, and lemon (i.e. 5 types)

How many different donut boxes can you buy?

So, our problem has been reduced to how many ways are there are assign these 4 dividers among the 12 donuts?

3 chocolate, 4 strawberry, 0 coconut, 2 blueberry, 3 lemon

Donuts

You're going to buy one-dozen donuts (i.e., 12 donuts) from chocolate, strawberry, coconut, blueberry, and lemon (i.e. 5 types)

How many different donut boxes can you buy?

1. Define an arbitrary ordering of the flavors (order doesn't matter, so count only 1)
   in our example, our ordering is chocolate, strawberry, coconut, blueberry, lemon
2. Place bars between the donuts to demarcate flavor

So, our problem has been reduced to how many ways are there are assign these 4 dividers among the 12 donuts?

3 chocolate, 4 strawberry, 0 coconut, 2 blueberry, 3 lemon

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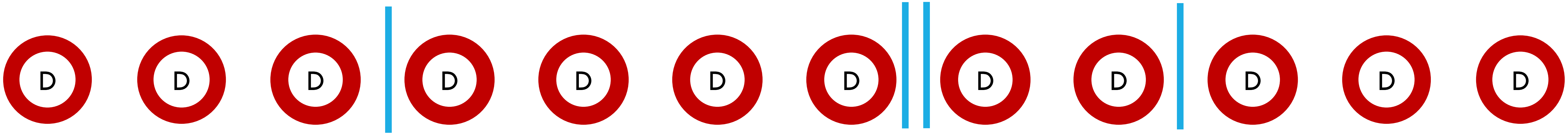

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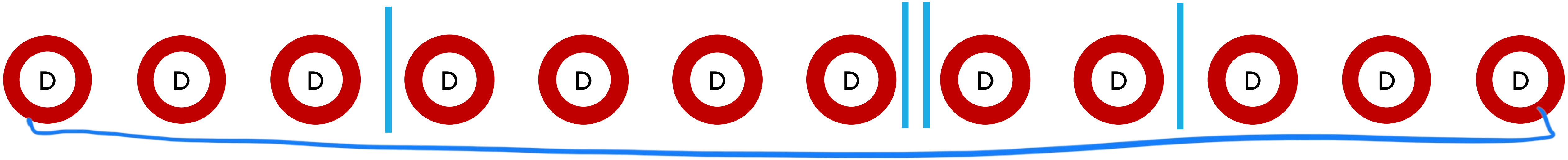


Approach 1:

This is a string rearranging problem! We have a string with 12 identical D's (donuts) and 4 identical |'s (dividers) and are rearranging that word.

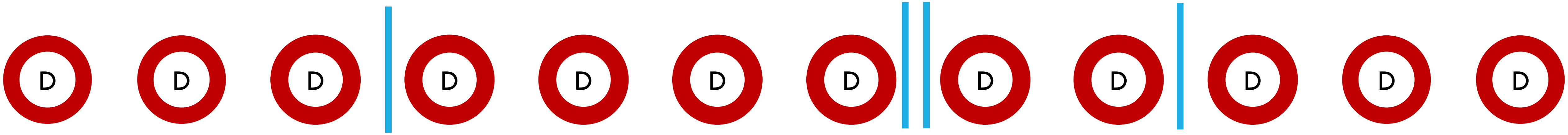
e.g., "DDD|DDDD||DD|DDD", DDDDDDDDDDDD||||

$\binom{16}{4}$

$\frac{16!}{4!\,12!}$

Donuts

So, our problem has been reduced to how many ways are there are assign these 4 dividers among the 12 donuts?



Approach 1:

This is a <u>string rearranging problem</u>! We have a string with 12 identical D’s (donuts) and 4 identical |’s (dividers) and are rearranging that word.

e.g., “DDD|DDDD||DD|DDD”, DDDDDDDDDDDD||||

1. Arrange letters in the string: $(12+4)! = 16!$

2. Divide out overcounting for duplicate letters: $\frac{16!}{12!4!} = \binom{16}{12}$

Donuts

So, our problem has been reduced to how many ways are there are assign these 4 dividers among the 12 donuts?

— — — — — — — — — — — — — — — —

Approach 2:

There are $12 + 4 = 16$ positions in total. 4 of these will be a divider and the remaining 12 will be donuts

Donuts

So, our problem has been reduced to how many ways are there are assign these 4 dividers among the 12 donuts?

Approach 2:

There are $12 + 4 = 16$ positions in total. 4 of these will be a divider and the remaining 12 will be donuts

1. Pick 12 of the 16 positions to be donuts: $\binom{16}{12}$

1, 2, 3 . . . .
2, 1, 3 . .

Donuts

So, our problem has been reduced to how many ways are there are assign these 4 dividers among the 12 donuts?

Approach 2:

There are $12 + 4 = 16$ positions in total. 4 of these will be a divider and the remaining 12 will be donuts

1. Pick 12 of the 16 positions to be donuts: $\binom{16}{12}$
2. Pick 4 of the remaining 4 positions to be donuts: $\binom{4}{4} = 1$

Donuts

So, our problem has been reduced to how many ways are there are assign these 4 dividers among the 12 donuts?

Approach 2:

There are $12 + 4 = 16$ positions in total. 4 of these will be a divider and the remaining 12 will be donuts

1. Pick 12 of the 16 positions to be donuts: $\binom{16}{12}$
2. Pick 4 of the remaining 4 positions to be ~~donuts~~ dividers: $\binom{4}{4} = 1$

$\binom{16}{12}$ ways in total – same as with previous approach!

In General

Stars and Bars

To pick n objects from k groups (where **order doesn't matter** and every **element of each group is indistinguishable**), use the formula:

$$\binom{n+k-1}{n} = \binom{n+k-1}{k-1}$$

This counting technique is often called "stars and bars"
using a "star" instead of a donut shape, and calling the dividers "bars"

> What often hints to use the stars-and-bars approach is when we're dealing with picking a set from objects where objects from the same types are **indistinguishable**

Proofs Using Counting Techniques

Combinatorial Proofs

Let’s first look at some combination facts…

Reminder - k-combination

k-combination

The number of k-element subsets from a set of n symbols is:

$$C(n,k) = \binom{n}{k} = \frac{P(n,k)}{k!} = \frac{n!}{k!\,(n-k)!}$$

Some Facts about combinations

Here are some known facts about combinations:

> Symmetry of combinations: $\boxed{\binom{n}{k} = \binom{n}{n-k}}$

> Pascal's Rule: $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

Some Facts about combinations

Here are some known facts about combinations:

> Symmetry of combinations: $\binom{n}{k} = \binom{n}{n-k}$

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How do we prove these equations are true?

First Proof of Symmetry ($\binom{n}{k} = \binom{n}{n-k}$)

Proof 1: By algebra

$\binom{n}{k} = \frac{n!}{k!(n-k)!}$ Definition of Combination

$= \frac{n!}{(n-k)!k!}$ Algebra (commutativity of multiplication)

$= \binom{n}{n-k}$ Definition of Combination

First Proof of Symmetry ($\binom{n}{k} = \binom{n}{n-k}$)

Wasn’t that a great proof.

Airtight. No disputing it.

Got to say “commutativity of multiplication.”

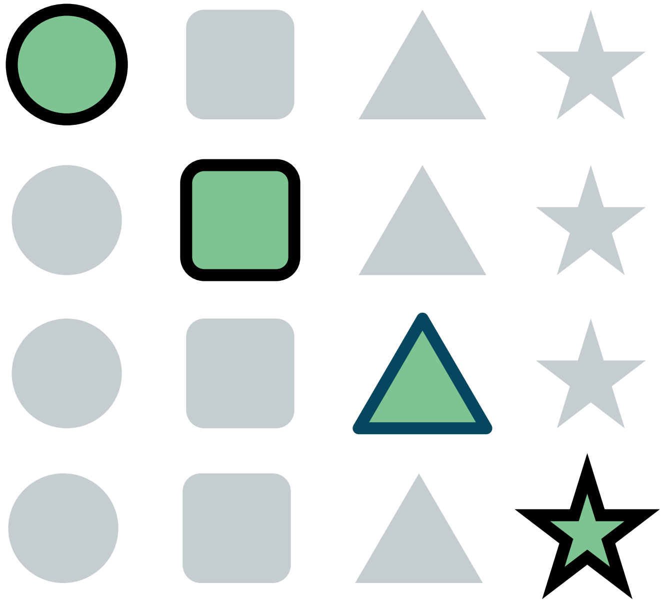
But…do you know *why*? Can you *feel* why it’s true?

Second Proof of Symmetry ($\binom{n}{k} = \binom{n}{n-k}$)

Both sides of this count the same set of outcomes! For example....

LHS: choose k things to include in the set

$\binom{4}{1} \rightarrow$



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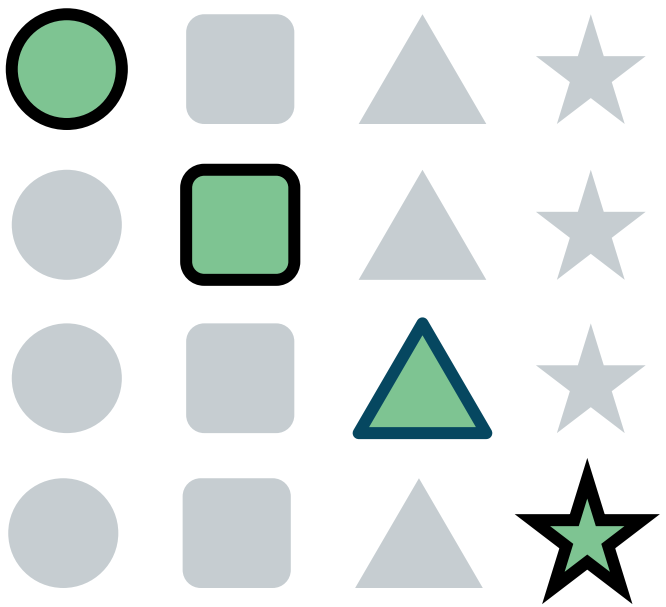
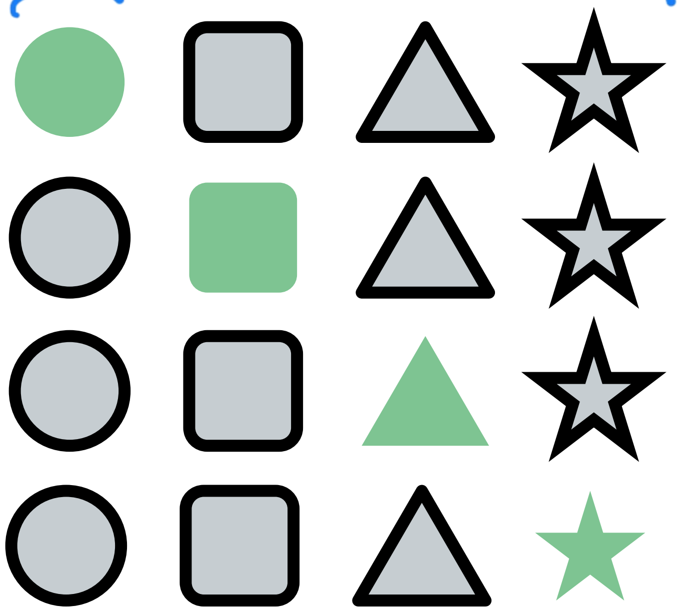


RHS: choose $\mathrm{n} - k$ things to <u>not</u> include in the set

$\binom{4}{3} \rightarrow$



Second Proof of Symmetry ($\binom{n}{k} = \binom{n}{n-k}$)

more formal – <u>combinatorial proof</u>

Suppose you have n people, and need to choose k people to be on your team. We will <u>count the number of possible teams two different ways</u>.

Way 1 (LHS): We choose the k people to be on the team. Since order doesn't matter (you're on the team or not), there are $\binom{n}{k}$ possible teams.

Second Proof of Symmetry ($\binom{n}{k} = \binom{n}{n-k}$)

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Suppose you have n people, and need to choose k people to be on your team. We will count the number of possible teams two different ways.

Way 1 (LHS): We choose the k people to be on the team. Since order doesn't matter (you're on the team or not), there are $\binom{n}{k}$ possible teams.

Way 2 (RHS): We choose the $n - k$ people to NOT be on the team. Everyone else is on it. Since order again doesn't matter, there are $\binom{n}{n-k}$ possible ways to choose the team.

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Way 2 (RHS): We choose the $n - k$ people to NOT be on the team. Everyone else is on it. Since order again doesn't matter, there are $\binom{n}{n-k}$ possible ways to choose the team.

Since we're counting the same thing, the numbers must be equal. So $\binom{n}{k} = \binom{n}{n-k}$.

Pascal's Rule: $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

algebraic proof

$$\begin{aligned}
\binom{n-1}{k-1} + \binom{n-1}{k} &= \frac{(n-1)!}{(k-1)!(n-1-[k-1])!} + \frac{(n-1)!}{k!(n-1-k)!} && \text{definition of combination} \\
&= \frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)!}{k!(n-k-1)!} && \text{subtraction} \\
&= \frac{[(n-1)!k!(n-k-1)!] + [(n-1)!(k-1)!(n-k)!]}{k!(k-1)!(n-k)!(n-k-1)!} && \text{Find a common denominator} \\
&= \frac{(n-1)!(k-1)!(n-k-1)!\,[k+(n-k)]}{k!(k-1)!(n-k)!(n-k-1)!} && \text{factor out common terms} \\
&= \frac{(n-1)!\,[k+(n-k)]}{k!(n-k)!} && \text{Cancel } (k-1)!(n-k-1)! \\
&= \frac{(n-1)!\cdot n}{k!(n-k)!} = \frac{n!}{k!(n-k)!} && \text{Algebra} \\
&= \binom{n}{k} && \text{Definition of combination}
\end{aligned}$$

Pascal's Rule: $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

combinatorial proof

Suppose you have n people (one of which is Michael), and need to choose k people to be on your team.

Pascal's Rule: $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ ***combinatorial proof***

Suppose you have n people (one of which is Michael), and need to choose k people to be on your team.

Way 1: There are n people total, of which we're choosing k (and since it's a team order doesn't matter) $\binom{n}{k}$.

Pascal's Rule: $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

sum rule cases

combinatorial proof

Suppose you have n people (one of which is Michael), and need to choose k people to be on your team.

Way 1: There are n people total, of which we're choosing k (and since it's a team order doesn't matter) $\binom{n}{k}$.

Way 2: There are two types of teams. Those for Michael makes the team, and those for Michael does not.

If Michael does make the team, then $k-1$ of the other $n-1$ also make it.

If Michael doesn't make the team, k of the other $n-1$ are on the team.

Overall, by sum rule, $\binom{n-1}{k-1} + \binom{n-1}{k}$.

Since we're computing the same number two different ways, they must be equal. So: $\boxed{\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}}$

Combinatorial Proofs

1. Describe a scenario
2. Explain how the LHS counts the outcomes in that scenario
3. Explain how the RHS counts the outcomes in that same scenario
4. State that "because the LHS and RHS both count the number of outcomes in the same scenario, they must be equal"

Combinatorial Proofs

When to use?
> Algebraic proofs are often difficult and don't give us an "intuitive" reason for why the equation holds
> *Combinatorial proofs* are used for proving an equation that may involve combinations, factorials, permutations, etc.

1. Describe a scenario
2. Explain how the LHS counts the outcomes in that scenario
3. Explain how the RHS counts the outcomes in that <u>same</u> scenario
4. State that "because the LHS and RHS both count the number of outcomes in the same scenario, they must be equal"

Combinatorial Proofs – tips!

> Start with the side that "looks simpler"

> Multiplying terms together indicates some sequential process (the product rule)

> A summation/adding terms indicates some disjoint cases that are added together using the sum rule

Pigeonhole Principle

A counting property that will be helpful in many proofs!

A Hairy Question

Are there at least two people in Seattle with the same number of hairs on their head?

A. Yes

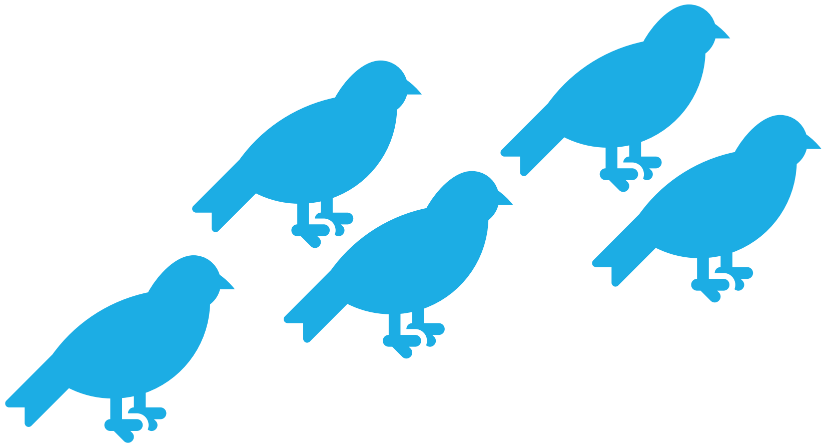
B. No

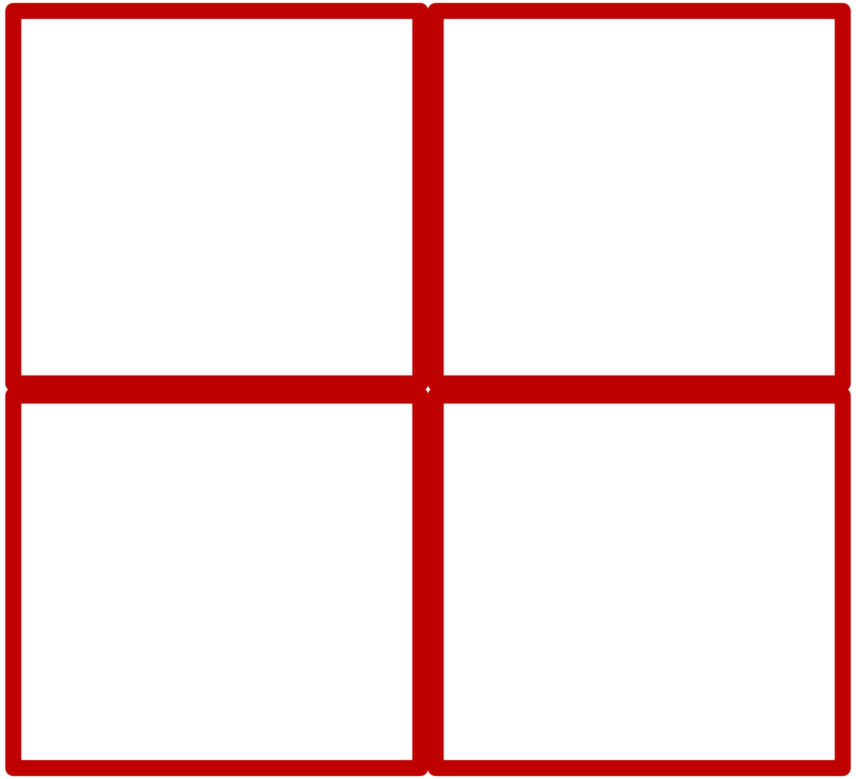
C. How would I know?

Go to **pollev.com/cse312**
(login with your UW identity)

Pigeonhole Principle

If you have 5 pigeons, and place them into 4 holes, then…

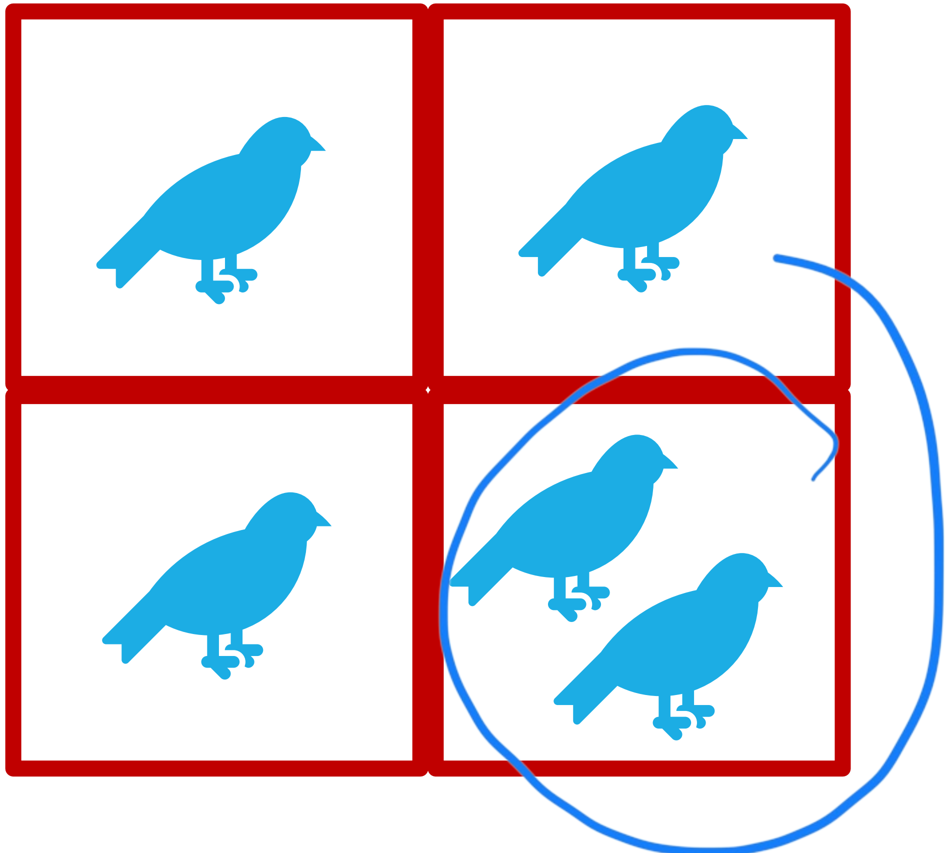




Pigeonhole Principle

If you have 5 pigeons, and place them into 4 holes, then…

At least 2 pigeons are in the same hole.



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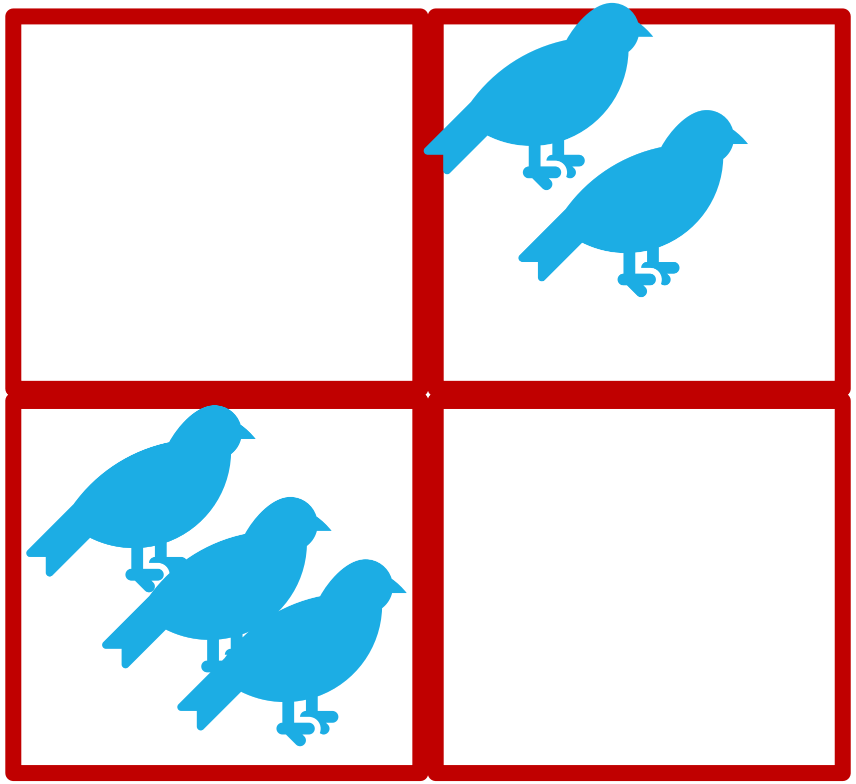

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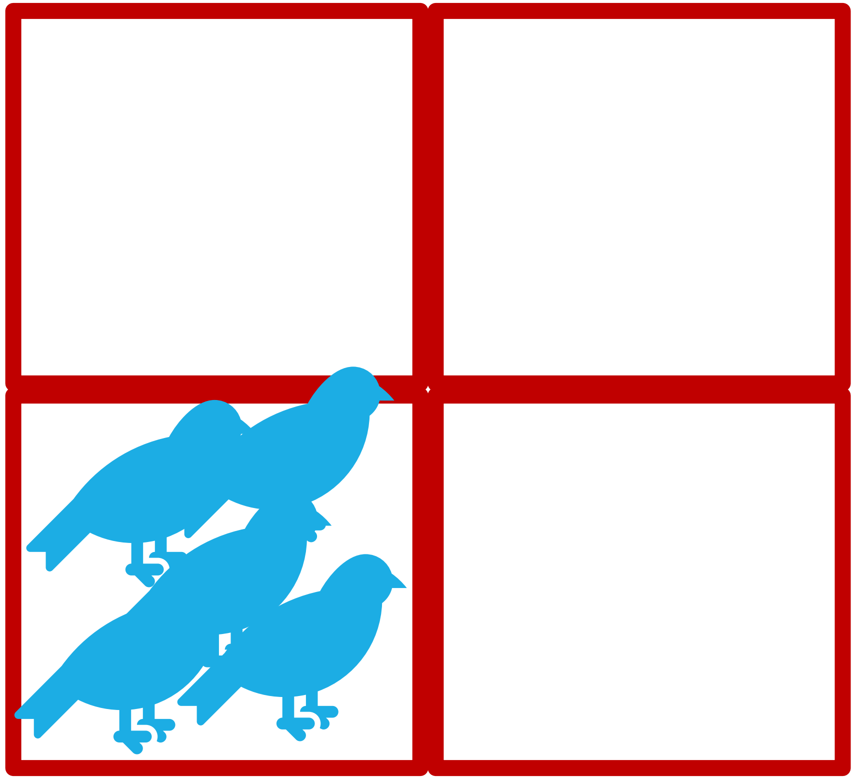


Pigeonhole Principle

Pigeonhole Principle

Version 1

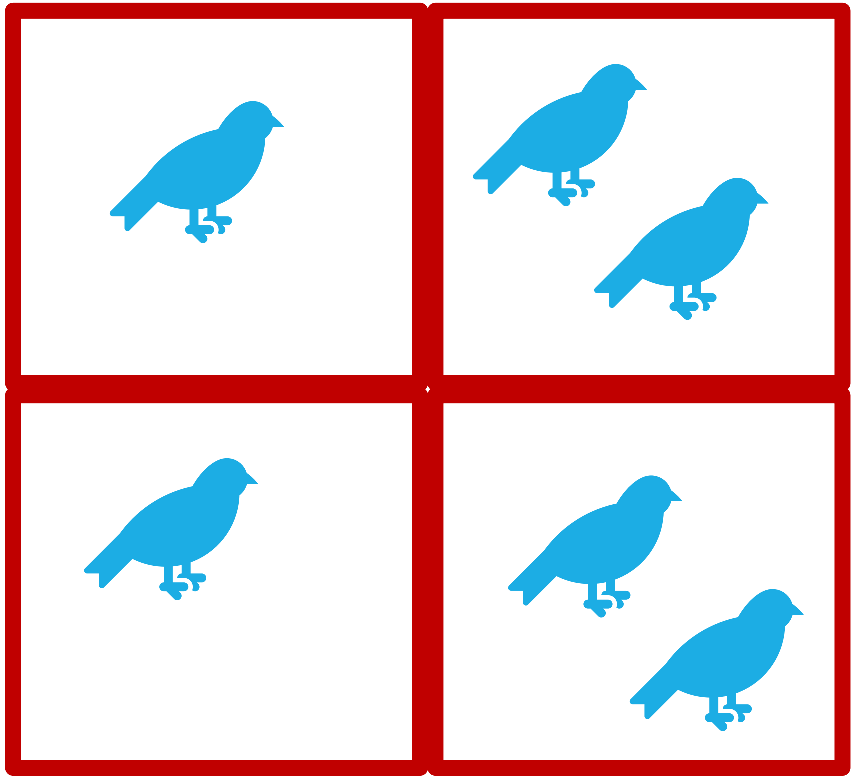
If you have $\boldsymbol{n}$ pigeons, and place them into $\boldsymbol{k}$ holes, and $\boldsymbol{n} > \boldsymbol{k}$, then **at least** 2 pigeons are in the same hole.

Note: we're not making guarantees about every pigeonhole – we're just guaranteeing that there is <u>at least one hole</u> with <u>at least</u> a certain number of pigeons

Pigeonhole Principle

If you have **6** pigeons, and place them into 4 holes, then…

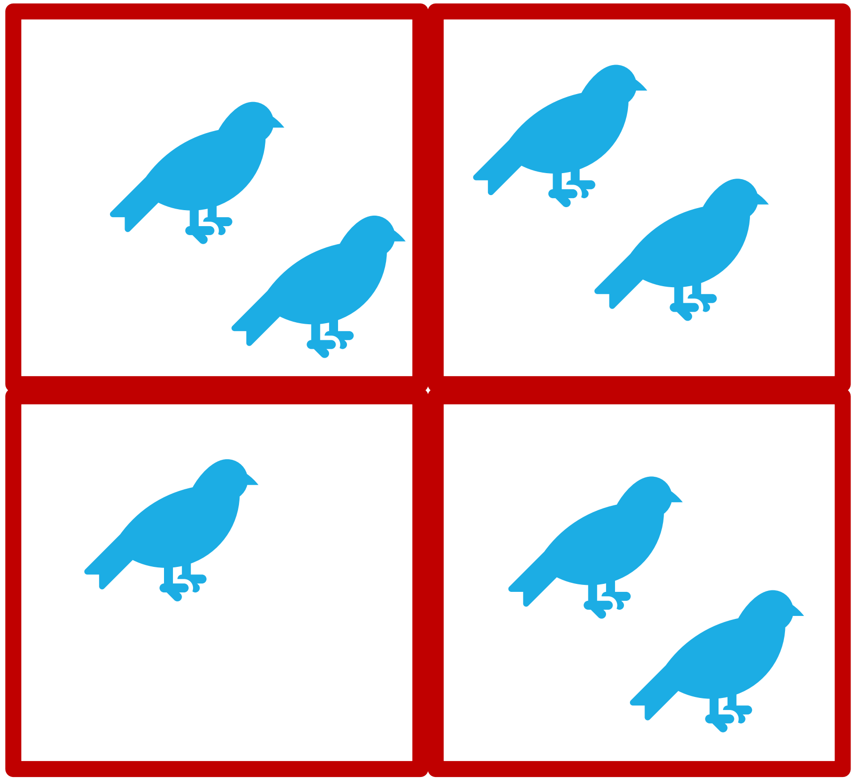
At least 2 pigeons are in the same hole.



Pigeonhole Principle

If you have **7** pigeons, and place them into 4 holes, then…

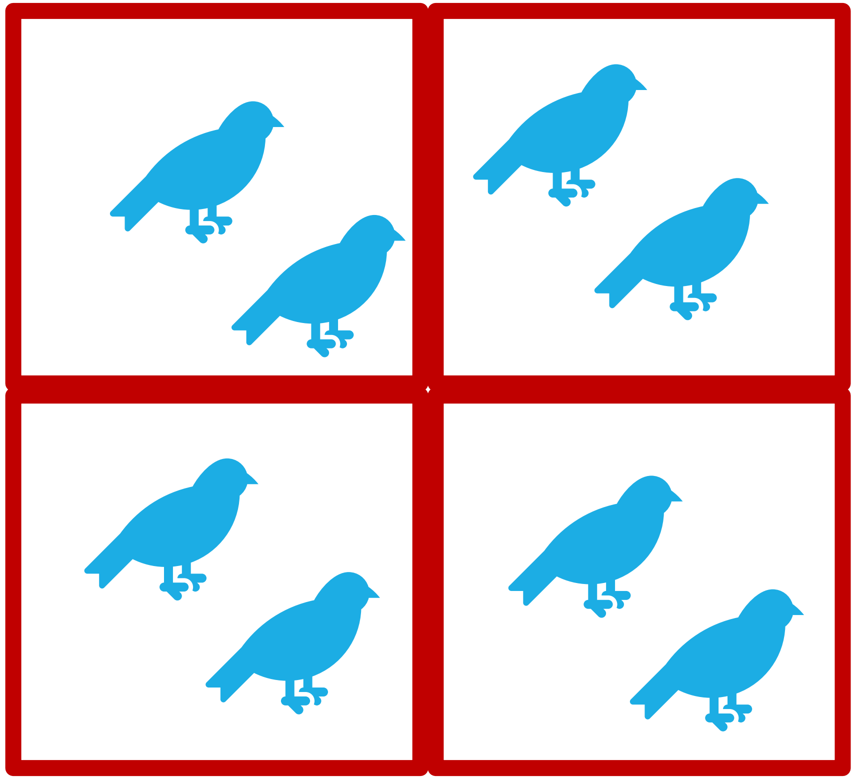
At least 2 pigeons are in the same hole.



Pigeonhole Principle

If you have **8** pigeons, and place them into 4 holes, then…

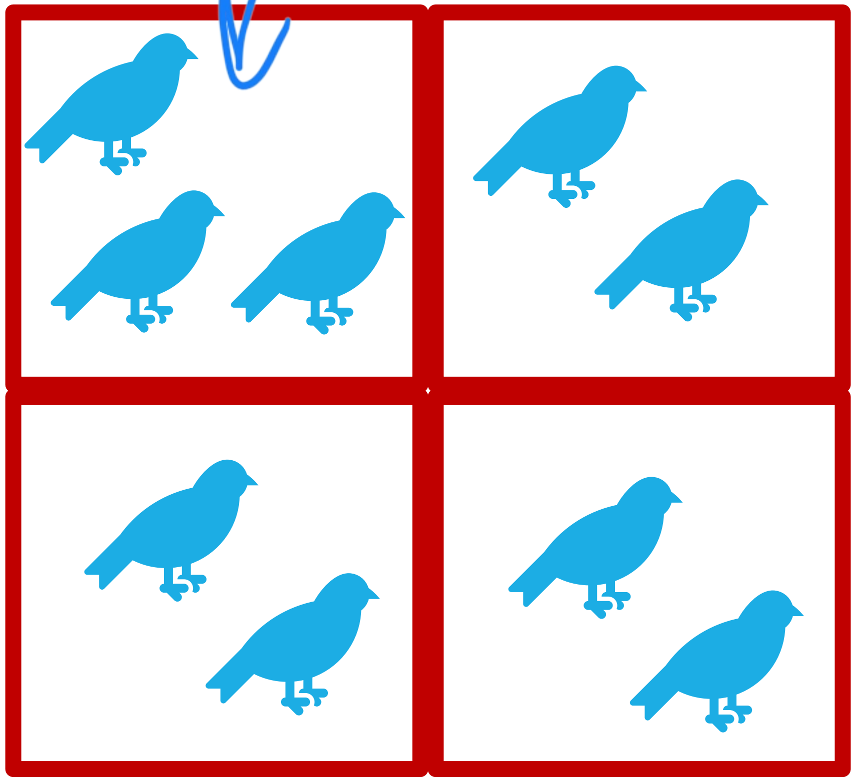
At least 2 pigeons are in the same hole.



Pigeonhole Principle

If you have **9** pigeons, and place them into **4** holes, then…

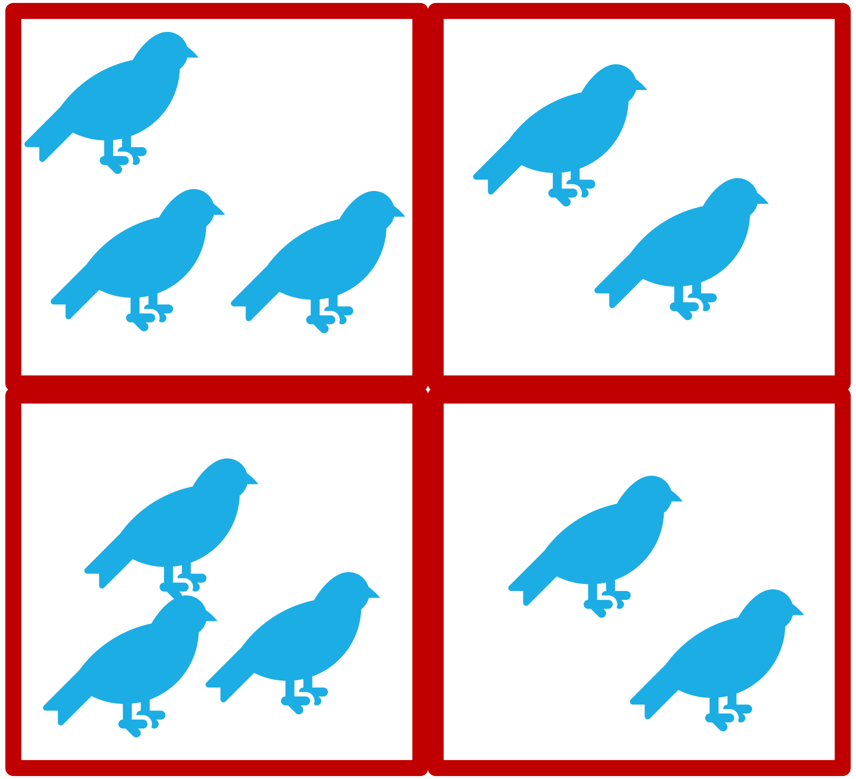
At least 3 pigeons are in the same hole.



Pigeonhole Principle

If you have $\mathbf{10}$ pigeons, and place them into 4 holes, then…

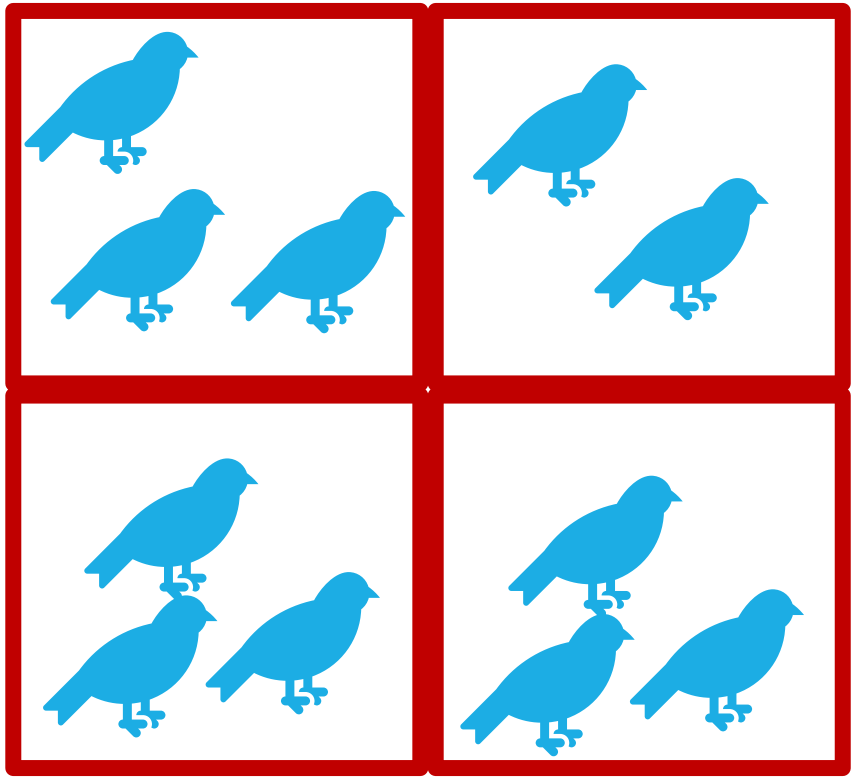
At least 3 pigeons are in the same hole.



Pigeonhole Principle

If you have **11** pigeons, and place them into 4 holes, then…

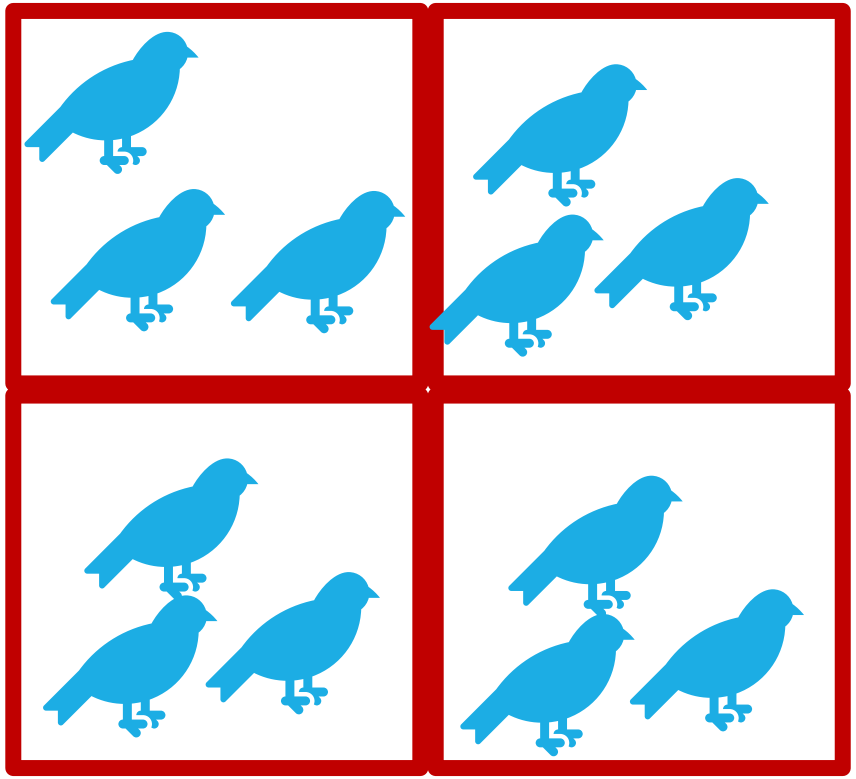
At least 3 pigeons are in the same hole.



Pigeonhole Principle

If you have **12** pigeons, and place them into **4** holes, then…

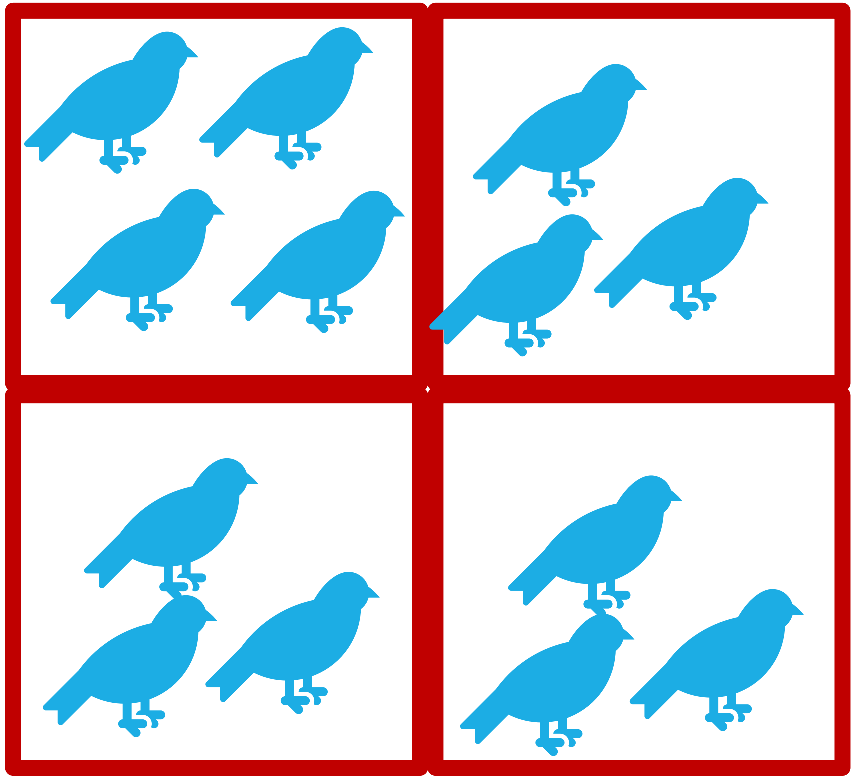
At least 3 pigeons are in the same hole.



Pigeonhole Principle

If you have **13** pigeons, and place them into 4 holes, then…

At least 4 pigeons are in the same hole.



Pigeonhole Principle

Pigeonhole Principle

Version 1

If you have $\boldsymbol{n}$ **pigeons**, and place them into $\boldsymbol{k}$ **holes**, and $\boldsymbol{n} > \boldsymbol{k}$, then **at least** 2 pigeons are in the same hole.

Version 2 (generalized)

If you have $\boldsymbol{n}$ **pigeons**, and place them into $\boldsymbol{k}$ **holes**, and $\boldsymbol{n} > \boldsymbol{k}$, then **at least** $\left\lceil \frac{n}{k} \right\rceil$ pigeons are in the same hole.

Note: we're not making guarantees about every pigeonhole – we're just guaranteeing that there is <u>at least one hole</u> with <u>at least</u> a certain number of pigeons

Pigeonhole Principle (generalized)

If you have n pigeons and k pigeonholes, then there is at least one pigeonhole that has at least $\left\lceil \frac{n}{k} \right\rceil$ pigeons.

$\lceil \ \rceil$

$\lceil a \rceil$ is the "ceiling" of a (it means always round up, $\lceil 1.1 \rceil = 2$, $\lceil 1 \rceil = 1$).

e.g., 5 pigeons and 4 holes -> at least 1 hole has $\left\lceil \frac{5}{4} \right\rceil = \lceil 1.25 \rceil = 2$ *pigeons*

Pigeonhole Principle (generalized)

If you have n pigeons and k pigeonholes, then there is <u>at least one</u> pigeonhole that has at least $\left\lceil \frac{n}{k} \right\rceil$ pigeons.

$\lceil a \rceil$ is the "ceiling" of a (it means always round up, $\lceil 1.1 \rceil = 2$, $\lceil 1 \rceil = 1$).

e.g., 13 pigeons and 4 holes -> at least 1 hole has $\left\lceil \frac{13}{4} \right\rceil = \lceil 3.25 \rceil = 4$ *pigeons*

An example

If you have to take 10 classes, and have 3 quarters to take them in, then…

Pigeons: The classes to take 10

Pigeonholes: The quarter 3

Mapping: Which class you take the quarter in
(each class will be assigned to exactly 1 quarter)

Applying the pigeonhole principle, there is at least one quarter where you take at least $\left\lceil \frac{10}{3} \right\rceil = 4$ courses.

An example

If you have to take 10 classes, and have 3 quarters to take them in, then…

Pigeons: The classes to take

Pigeonholes: The quarter

Mapping: Which class you take the quarter in
(each class will be assigned to exactly 1 quarter)

Remember!
When defining pigeons and pigeonholes, every pigeon **must** be assigned to one pigeonhole

Applying the pigeonhole principle, there is at least one quarter where you take at least $\left\lceil \frac{10}{3} \right\rceil = 4$ courses.

Practical Tips

1. What are the **pigeons?**
2. What are the **pigeonholes**?
3. How do you **map from pigeons to pigeonholes**?
4. There are n pigeons and k pigeonholes, so **by pigeonhole principle**, there is at least 1 pigeonhole with at least $\left\lceil \frac{n}{k} \right\rceil$ pigeons. This is what we're trying to prove because…

Practical Tips

When to use pigeonhole principle?

> When we're trying to **prove** or **make some guarantee** about a certain number of things sharing some property.

> We'll usually warn you in advance that pigeonhole principle the right method (you'll see one in the section handout).

1. What are the **pigeons?**

2. What are the **pigeonholes?**

3. How do you **map from pigeons to pigeonholes?**

4. There are n pigeons and k pigeonholes, so **by pigeonhole principle**, there is at least 1 pigeonhole with at least $\left\lceil \frac{n}{k} \right\rceil$ pigeons. This is what we're trying to prove because…

Practical Tips

Look for – a set you're trying to divide into groups, where collisions would help you somehow.

"Prove that there are at least **x A's** that have the same **B"**

> Pigeons: all possibles A's

> Pigeonholes: all possible B's

> Mapping: each A is assigned to one possible B

The pigeonhole principle tells us that there are at least x pigeons that fall into the same hole -> there are at least x A's that have the same B

Back to the hairy question….

Are there at least two people in Seattle with the same number of hairs on their head?

Pigeons: people in Seattle

Pigeonholes: # hairs on head

Mapping:

Back to the hairy question….

Are there at least two people in Seattle with the same number of hairs on their head?

Pigeons: All people in Seattle

750,000 people in Seattle (**750,000 pigeons**)

Pigeonholes: All possible 'hair counts' -> ~

people have up to 150,000 hairs on their head (**150,000 pigeonholes**)

Mapping: Every person is "assigned" to a particular hair count

By the **pigeonhole principle**, there are at least $\left\lceil \frac{750,000}{150,000} \right\rceil = 5$ pigeons in the same pigeonhole -> <u>at least 5 people in Seattle with the same hair count!</u>

But actually…what's the point of this?

- We can prove some pretty fun statements!
- Hashing Algorithms: Ensuring that collisions will occur when hashing more items than there are available hash slots, which helps in designing effective collision resolution strategies. *(you'll learn about hashing in 332!)*
- Resource Allocation: Ensuring that in systems with more requests than resources, some resources will be shared (e.g., tasks and processors)
- Data Compression: Demonstrating that lossless compression of sufficiently large files must lead to some files being larger after compression due to limited encoding space
- It is also the basis of proving that a language is not regular, as you may remember from 311!

Thinking about Counting

We've seen lots of ways to count

Sum rule (split into disjoint sets) cases

Product rule (use a sequential process) $a \cdot b \cdot \ldots$

Combinations (order doesn't matter) $\{a, b, c\}$ pick k from n

Permutations (order does matter) aba, cba, ...

Principle of Inclusion-Exclusion (counting the size of a union of sets) $|A \cup B|$

Complementary Counting (counting the ways for something to not occur)

"Stars and Bars" $\binom{n+k-1}{k-1}$ (assigning indistinguishable stars to distinguishable bins)

Niche Rules (useful in very specific circumstances)

- Binomial Theorem $(x+y)^n$
- Pigeonhole Principle

How to tell which approach(es) to take?

order

- Identify keywords in the problem, and key properties, and think about what techniques match those patterns
- If an approach isn't working or things are getting out of control, try a different approach!
- There are often multiple ways to solve the same problem ☺
- In some harder problems, we might need to start by solving a simplified version of the problem and then dividing/subtracting out overcounting (we'll see an example of this now!)

This all takes practice so don't be hard on yourself if you don't think of the correct answer at first glance!!

How to tell which approach(es) to take?

> Can we break this into some disjoint cases? (**sum rule**)

> Can we create our desired outcomes with a clear sequential process? (**product rule**)

> Are we counting the ways for something <u>not</u> to occur? (**complementary counting**)

> Are we dealing with the union of some sets? (**inclusion-exclusion**)

> Does order matter or not in this problem? (**permutation** or **combination**)

> Are the objects distinguishable or indistinguishable (**stars and bars** if indistinguishable)

These approaches often work together! E.g., to compute the number of options in one of the disjoint cases, you may need another approach

More examples

Putting it all together!

Cards

A lot of counting problems deal with cards!

A "standard" deck of cards has 52 cards ($13 \cdot 4 = 52$).

Each card has one of **4 suits**

diamonds ♦,

hearts ♥,

clubs ♧,

spades ♠

and one of **13 values/ranks** (Ace,2,3,4,5,6,7,8,9,10,Jack,Queen, King).

e.g., Ace♦, 5♦, 5♧, 10♠ are all possible cards

A "k-card-hand" is an unordered set of k cards

5-card-hand

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e.g., **Ace♦, 5♦, 5♧, 10♠** are all possible cards

A "k-card-hand" is an unordered set of k cards

How many five-card "flushes" are there? – a flush is a hand of cards all of the same suit. (e.g., {A♠, 3♠, 5♠, 6♠, Q♠})

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Five-card "flushes"

How many five-card "flushes" are there? – a flush is a hand of cards all of the same suit.

Think: How would I create a set of cards that is a flush?

Way 1:

1. Pick the suit (e.g., ♠) – $\binom{\mathbf{4}}{\mathbf{1}}$

2. Pick the specific values/cards from that suit (e.g., {A,3,5,6,Q}) - $\binom{\mathbf{13}}{\mathbf{5}}$

Now we've created an unordered 5-card flush! (e.g., {A♠, 3♠, 5♠, 6♠, Q♠})

$\boxed{\binom{4}{1} \cdot \binom{13}{5}}$

Five-card "flushes"

Way 2:

Pretend order matters.

1. Pick any first card – 52 options

2. All remaining cards must be from the same suit of that first suit:

   12 options for the 2^{nd} card, 11 options for the 3^{rd} card, etc.

Divide out the overcounting - divide by $5!$, since order isn't supposed to matter (i.e., only count each unordered flush once)

$$\boxed{\frac{52 \cdot 12 \cdot 11 \cdot 10 \cdot 9}{5!}}$$

This equals the same number as what we got on the last slide!

How many 5-card hands have <u>at least</u> 3 aces?

There are 4 Aces (and 48 non aces) in a deck of cards

1. **Choose $\mathbf{3}$ aces**: $\binom{4}{3}$

2. Then **pick $\mathbf{2}$ of the $\mathbf{49}$ remaining cards** to form a 5(the last ace is allowed as well, because we're allowed to have all 4): $\binom{49}{2}$

$\binom{4}{3} \cdot \binom{49}{2}$

What's wrong with this calculation? Does it,

A) Overcount B) Undercount C) It's correct! D) I have no idea :)

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Sleuth’s Criterion

How to check if we counted correctly?

For each outcome that we want to count, there should be exactly one set of choices in the sequential process that will lead to that outcome.

> If there are no sequence of choices that will lead to the outcome, we have **undercounted**.

> If there is more than one sequence of choices that will lead to the outcome, we have **overcounted**.

Sleuth’s Criterion (in context)
How to check if we counted correctly?

For each “5-card hands with at least 3 aces” that we want to count, there should be exactly one set of choices in the sequential process that will lead to that outcome.

> If there are no sequence of choices that will lead to a particular 5-card hand with at least 3 aces, we have **undercounted**.

> If there is more than one sequence of choices that will lead to a particular **5-card hand with at least 3 aces**, we have **overcounted**.

How many 5-card hands have at least 3 aces?

For each "5-card hands with at least 3 aces" that we want to count, there should be exactly one set of choices in the sequential process that will lead to that outcome.

A♧, A♠, A♦, Q♥, K ♠ is a valid outcome should counted exactly once.

Step 1 (choose 3 aces): {A♧, A♠, A♦}
Step 2 (pick 2 of remaining 49): {Q♥, K ♠}

Great! There's no other set of choices that will lead to this hand.

How many 5-card hands have at least 3 aces?

For each “5-card hands with at least 3 aces” that we want to count, there should be exactly one set of choices in the sequential process that will lead to that outcome.

A♧, A♠, A♦, A♥, K ♠ is a valid outcome should counted exactly once. But…

Step 1 (choose 3 aces): {A♧, A♠, A♦}
Step 2 (pick 2 of remaining 49): {A♥, K ♠}

How many 5-card hands have at least 3 aces?

For each "5-card hands with at least 3 aces" that we want to count, there should be exactly one set of choices in the sequential process that will lead to that outcome.

A♧, A♠, A♦, A♥, K ♠ is a valid outcome should counted exactly once. But…

Step 1 (choose 3 aces): {A♧, A♠, A♦}
Step 2 (pick 2 of remaining 49): {A♥, K ♠}

Step 1 (choose 3 aces): {A♧, A♥, A♦}
Step 2 (pick 2 of remaining 49): {A ♠, K ♠}

Both of these are different choices in the sequential process and are counted separately, but they are the same hand!
This is overcounting ☹

Fixing The Overcounting

How many 5-card hands have at least 3 aces?

Way 1: We could start with our incorrect solution & **subtract the**

Our original incorrect solution:

1. Choose **3** aces: $\binom{4}{3}$, 2. Then **pick 2 of the 49 remaining cards**: $\binom{49}{2}$ → $\binom{4}{3} \cdot \binom{49}{2}$

What kinds of hands do we overcount (counted many times in the sequential process)?

> 5-card hands with 4 Aces (i.e., a hand like {A♧, A♠, A♦, A♥, X})

How many 5-card hands have <u>at least </u>3 aces?

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Our original incorrect solution:
1. Choose **3** aces: $\binom{4}{3}$, 2. Then **pick 2 of the 49 remaining cards:** $\binom{49}{2}$ → $\binom{4}{3} \cdot \binom{49}{2}$

What kinds of hands do we overcount (counted many times in the sequential process)?

> 5-card hands with 4 Aces (i.e., a hand like {A♧, A♠, A♦, A♥, X})

> This hand is counted 4 different times (each row below is a different set of choices)

{A♧, A♠, A♦}, {A♥, X}

{A♧, A♠, A♥}, {A♦, X}

{A♧, A♥, A♦}, {A♠, X}

{A♥, A♠, A♦}, {A♧, X}

How many 5-card hands have <u>at least</u> 3 aces?

Way 1: We could start with our incorrect solution & **subtract the overcounting.**

Our original incorrect solution:
1. **Choose 3 aces:** $\binom{4}{3}$, 2. Then **pick 2 of the 49 remaining cards:** $\binom{49}{2}$→ $\binom{4}{3} \cdot \binom{49}{2}$

What kinds of hands do we overcount (counted many times in the sequential process)?

> 5-card hands with 4 Aces (i.e., a hand like {A♣, A♠, A♦, A♥, **X**})

{A♣, A♠, A♦}, {A♥, **X**}
{A♣, A♠, A♥}, {A♦, **X**}
{A♣, A♥, A♦}, {A♠, **X**}
{A♥, A♠, A♦}, {A♣, **X**}

So, how many outcomes are overcounted?

> There are $\binom{4}{4} \cdot 48 = 48$ 5-card hands with all 4 Aces

> Each of these hands is counted 4 times, but we only want to count it once

> So we've counted $(4-1) \cdot 48 = 3 \cdot 48$ processes that shouldn't count.

That would give a corrected total of $\binom{4}{3} \cdot \binom{49}{2} - 3 \cdot 48$

How many 5-card hands have <u>at least </u>3 aces?

Way 1: We could **subtract out the overcounting** - count exactly which hands are overcounted in our sequential process, and how many times each of those hands are overcounted, and subtract that from our initial count.

$$\binom{4}{3} \cdot \binom{49}{2} - 3 \cdot 52$$

Way 2: Try a different approach! The problem with our original solutions was trying to account for the "at least" - **come up with disjoint sets and count separately.**

Case 1: There are exactly 3 aces: $\binom{4}{3} \cdot \binom{48}{2}$

Case 2: There are exactly (all) 4 aces: $\binom{4}{4} \cdot \binom{48}{1}$

Applying the sum rule: $\binom{4}{3} \cdot \binom{48}{2} + \binom{4}{4} \cdot \binom{48}{1}$

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Does this overcount/undercount?

For a valid outcome, there should be exactly 1 set of choices leading to that outcome:

A♧, A♠, A♦, Q ♧, K ♠ - this will fall under the first case. The only possible set of choices leading to this is {A♧, A♠, A♦} in the 1st step and {Q ♧, K ♠} in the 2nd

How many 5-card hands have at least 3 aces?

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Case 2: There are exactly (all) 4 aces: $\binom{4}{4} \cdot \binom{48}{1}$

Applying the sum rule: $\binom{4}{3} \cdot \binom{48}{2} + \binom{4}{4} \cdot \binom{48}{1}$

Does this overcount/undercount?

For a valid outcome, there should be exactly 1 set of choices leading to that outcome:

A♧, A♠, A♦, A♥, K ♠ - this will fall under the second case. The only possible set of choices leading to this is {A♧, A♠, A♦, A♥} in the 1st step and {K ♠} in the 2nd

Another Problem!

A Fruit Problem

You have to choose 8 pieces of fruit. There are apples, oranges, and bananas. Fruits of the same type are indistinguishable.

You need to pick at most 2 apples and at least 1 banana. How many sets of fruit can you choose?

A Fruit Problem

You have to choose 8 pieces of fruit. There are apples, oranges, and bananas. Fruits of the same type are indistinguishable. You need to pick **at most** 2 apples and **at least** 1 banana. How many sets of fruit can you choose?

Divide into disjoint cases based on number of apples:

0 apples:

1 apple:

2 apples:

__ total (by sum rule)

A Fruit Problem

You have to choose 8 pieces of fruit. There are apples, oranges, and bananas. Fruits of the same type are indistinguishable. You need to pick **at most** 2 apples and **at least** 1 banana. How many sets of fruit can you choose?

Divide into disjoint cases based on number of apples:

0 apples: 1 to 8 bananas possible (8 options)

1 apple: 1 to 7 bananas possible (7 options)

2 apples: 1 to 6 bananas possible (6 options)

21 total (by sum rule)

A Fruit Problem (another approach)

You have to choose 8 pieces of fruit. There are apples, oranges, and bananas. Fruits of the same type are indistinguishable. You need to pick **at most** 2 apples and **at least** 1 banana. How many sets of fruit can you choose?

Divide into <u>disjoint cases</u> based on number of apples:

0 apples: Pick 1 banana, select banana/orange for remaining: $\binom{7+2-1}{2-1} = 8$

1 apple: Pick 1 banana, select banana/orange for remaining: $\binom{6+2-1}{2-1} = 7$

2 apples: Pick 1 banana, select banana/orange for remaining: $\binom{5+2-1}{2-1} = 6$

21 total (by sum rule)

A Fruit Problem (and *another* approach!)

You have to choose 8 pieces of fruit. There are apples, oranges, and bananas. Fruits of the same type are indistinguishable. You need to pick **at most** 2 apples and **at least** 1 banana. How many sets of fruit can you choose?

1. Pick out your first banana – 1 option because indistinguishable

2. Pick 7 fruits (at most 2 apples, can take apples oranges / bananas)

Complementary counting:

Total possibilities ignoring apple restriction: $\binom{7+3-1}{3-1}$ (7 pieces, 3 types)

Possibilities with ≥ 3 apples (these are the outcomes we want to **exclude**)

$\binom{4+3-1}{3-1}$. Pick 3 apples (1 options bc indistinguishable), pick 4 from 3 types

Total: $\binom{9}{2} - \binom{6}{2} = 36 - 15 = 21$

Takeaways

For donut-counting style problems with “twists”, it sometimes helps to “just throw the first few in the box” to get a problem that is exactly in the donut-counting framework.

There are often many ways to do the same problem! When you can do a problem two **very** different ways and get the same answer, you get much more confident in the answer.