More Counting!

CSE 312 23Su

Lecture 2

Fun Fact

There are $52! \approx 8 \cdot 10^{67}$ possible arrangements of a deck of 52 cards. So, when you shuffle a deck of cards, the resulting order has likely never been seen before!

Announcements

Homework 1 was released, due next Wednesday (6/26)

> All content for the homework will be covered by today's lecture

> Collaboration is encouraged (!) but follow collaboration policy (syllabus)
>
> Fill out study group finding form posted on Ed if you'd like to

> Justify your answers! We want to see your thought process

> Coding question

> We're here to help!
>
> Office hour schedule posted on the course website
>
> Edstem discussion board

Announcements

You’ll have 7 late days to use for the quarter.

At most 2 late days per assignment.

Late days are for “normal” things during the quarter

If you have an unusual or extended or extreme issue, please let us know.

The sooner you let us know, the more options we have for accommodations.

Where Are We?

Last time:

Sum and Product Rules

Complementary Counting

Permutations and Combination

Today:

More about combinations

Applications of combinations – path counting, binomial theorem

Inclusion-Exclusion

Recap: k-permutation

$\boldsymbol{k}$-permutation

The number of $\boldsymbol{k}$-element sequences of distinct symbols from a universe of $\boldsymbol{n}$ symbols is:

$$\boldsymbol{P(n,k) = n \cdot (n-1) \cdots (n-k+1) = \frac{n!}{(n-k)!}}$$

Examples:

- Ordering k people from a group of n people in a line
- Assigning k players from a team of n to k different positions (so order matters)
- Awarding gold, silver, and bronze medal among n participants - $P(n, 3)$

Picture Time 📸

How many ways to arrange 3 people from a group of 20 people (A,B,C,…) to stand in a line for a picture?

this means that ABC, CBA, BCD, etc. are all counted as different outcomes

Creating an ordered sequence of people with no repeats -> permutation

$P(20,3) = 20 \cdot 19 \cdot 18$ because we are creating an ordered sequence of 3 people from a larger group of 20 people

Picture Time 📸 (but change it slightly)

How many ways to select 3 people from a group of 20 people (A,B,C,…) to be in a picture, but we don't care about the order they stand in?

this means ABC, CDF, BCD, etc. are all counted as different outcomes, ***but*** *ABC, BCA,CBA, etc. are counted as the same outcome*

Same as before but now we do not care about order!

Looking for the number of possible **subsets** of 3 from the group of 20

Clever approach – count two ways

Let's go back to the first problem, where order mattered.

The total options was $P(20,3) = \frac{20!}{(20-3)!}$

How else could we get an ordered list? With this sequential process:

Step 1: Choose a subset of 3 people. Let's say there are x options for this

Step 2: Put the subset in order. $3!$ ways to arrange 3 people

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Step 1: Choose a subset of 3 people. Let's say there are x options for this

Step 2: Put the subset in order. $3!$ ways to arrange 3 people

Both approaches better give us the same number, so:

$$P(20,3) = \frac{20!}{(20-3)!} = x \cdot 3!$$

So the number of size-5 subsets of a size-20 set is:

$$x = \frac{P(20,3)}{3!} = \frac{20!}{(20-5)!\,5!}$$

Number of Subsets - k-combination

k-combination

The number of k-element subsets from a set of n symbols is:

$$C(n,k) = \frac{P(n,k)}{k!} = \frac{n!}{k!\,(n-k)!}$$

How many ways to select 5 people from a group of 20 people (A,B,C,…) to be in a picture, but we don't care about the order they stand in?

Here, we're only interested in the **unordered subsets**, so we use a **combination:**

$$C(20,5)$$

Another approach for deriving k-combinations

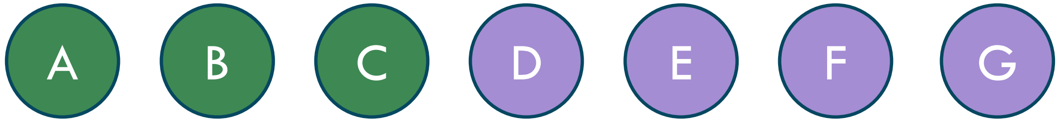
The second way of counting hints at a generally useful trick:

(1) Pretend that order does matter, then (2) divide by the number of orderings of the parts where order doesn't matter.

Here's another way to get the formula for combinations (choosing a subset of size k from a universe of n elements):

1. Put the n elements in some order. The first k are in the desired set. ($n!$)

e.g., n=7, k=3



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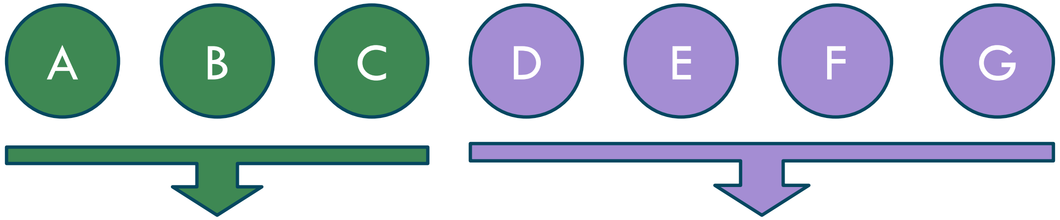


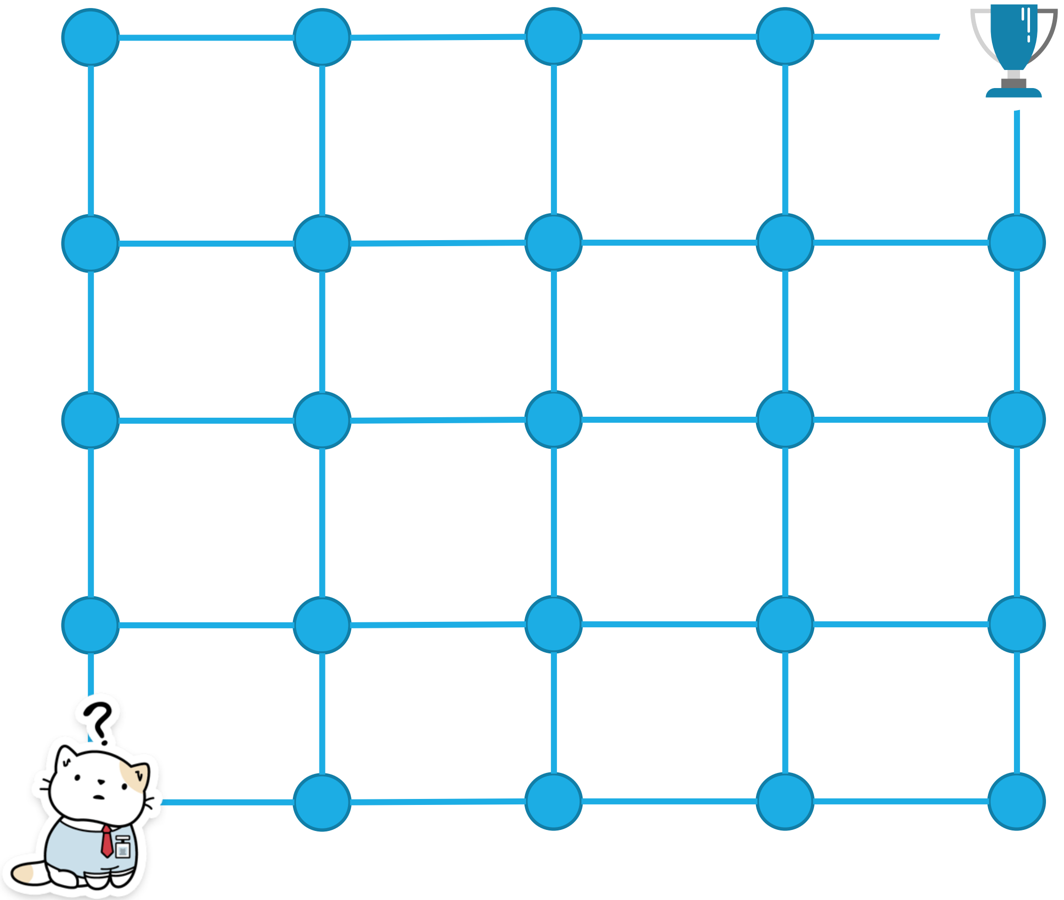
2. Among the first k, order doesn't matter between them. Divide by $k!$.

2. Among the last $n-k$, order doesn't matter between them. Divide by $(n-k)!$.

$$\boxed{\frac{n!}{k!\,(n-k)!}}$$

Path Counting 🏔️

Path Counting



We're in the lower-left corner, and want to get to the upper-right corner.

Each step either goes right or up.

How many different paths are there?

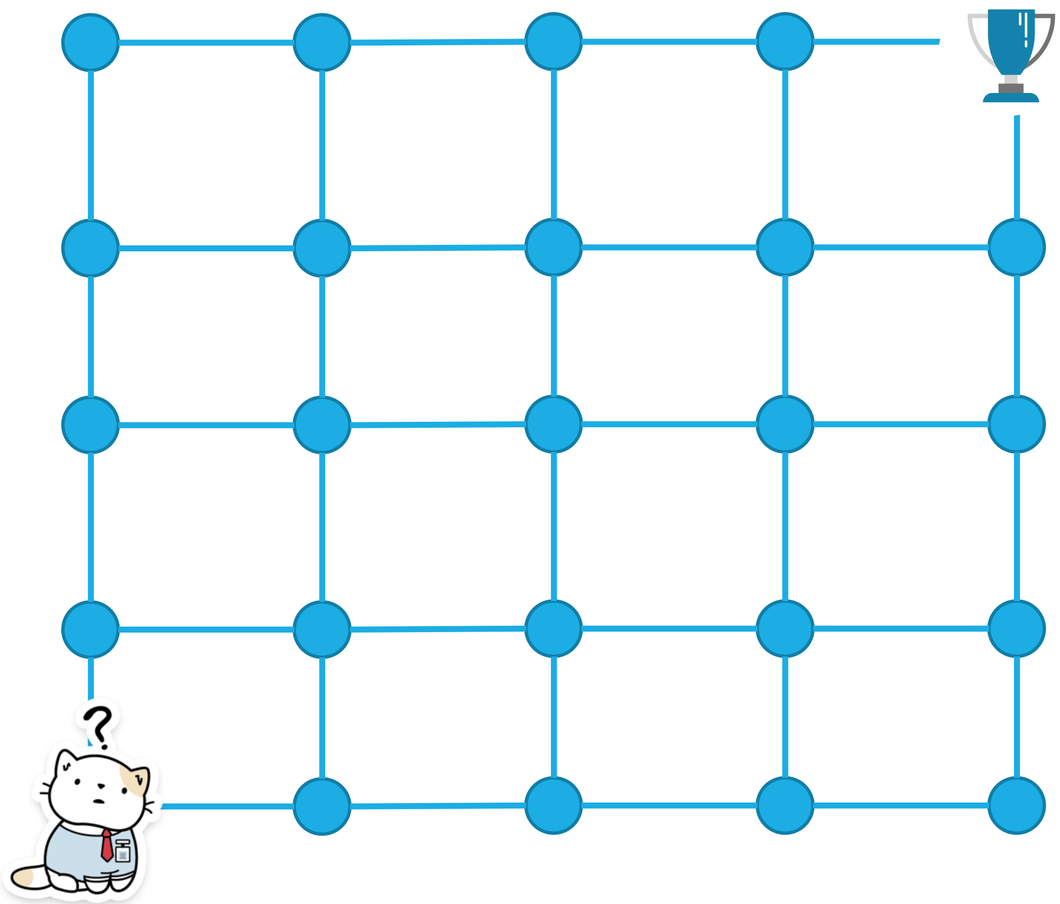
A. 2^8
B. $P(8,4)$
C. $\binom{8}{4}$
D. Something else

Fill out the poll everywhere so Claris knows how much to explain
Go to **pollev.com/cse312** and login with your UW identity

Path Counting

Go from lower-left corner to the upper-right corner.

Each step either goes right or up.



We're going to take 8 steps.

Maybe…2^8 options because on each step we either go right or up?

Move 1: (R/U) 2
Move 2:(R/U) 2
Move 3: (R/U) 2
Move 4: (R/U) 2
Move 5: (R/U) 2
Move 6: (R/U) 2
Move 7: (R/U) 2
Move 8: (R/U) 2

Path Counting

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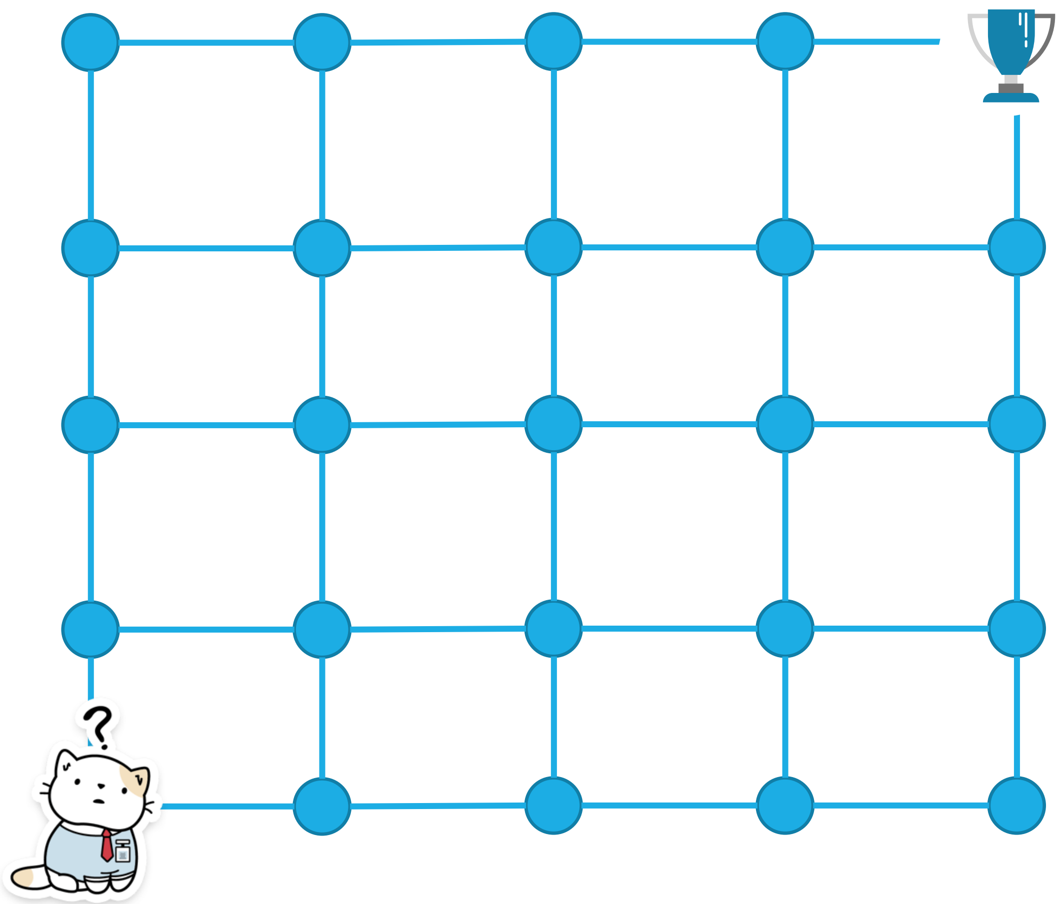

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Maybe…2^8 options because on each step we either go right or up?
But this includes paths that don't take us to the goal! ☹

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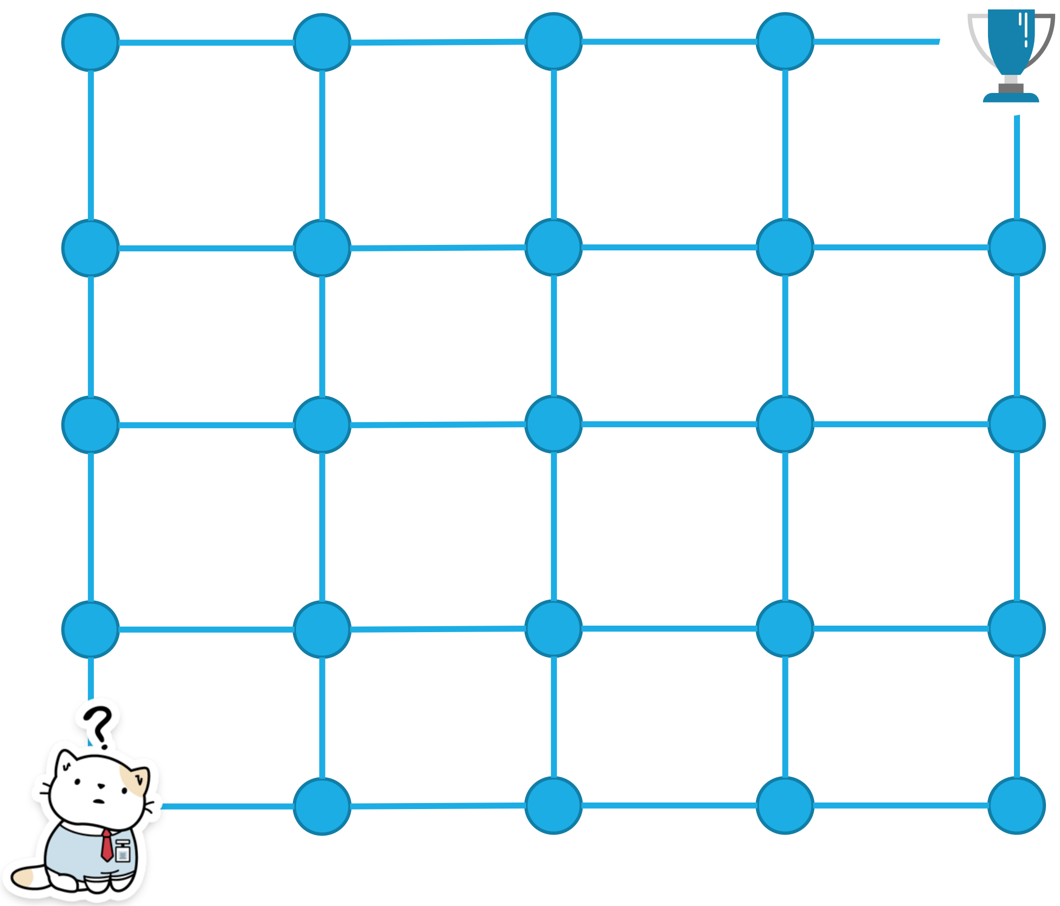

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We're going to take 8 steps.

Exactly 4 of the steps must be up and the rest to the right to reach the goal

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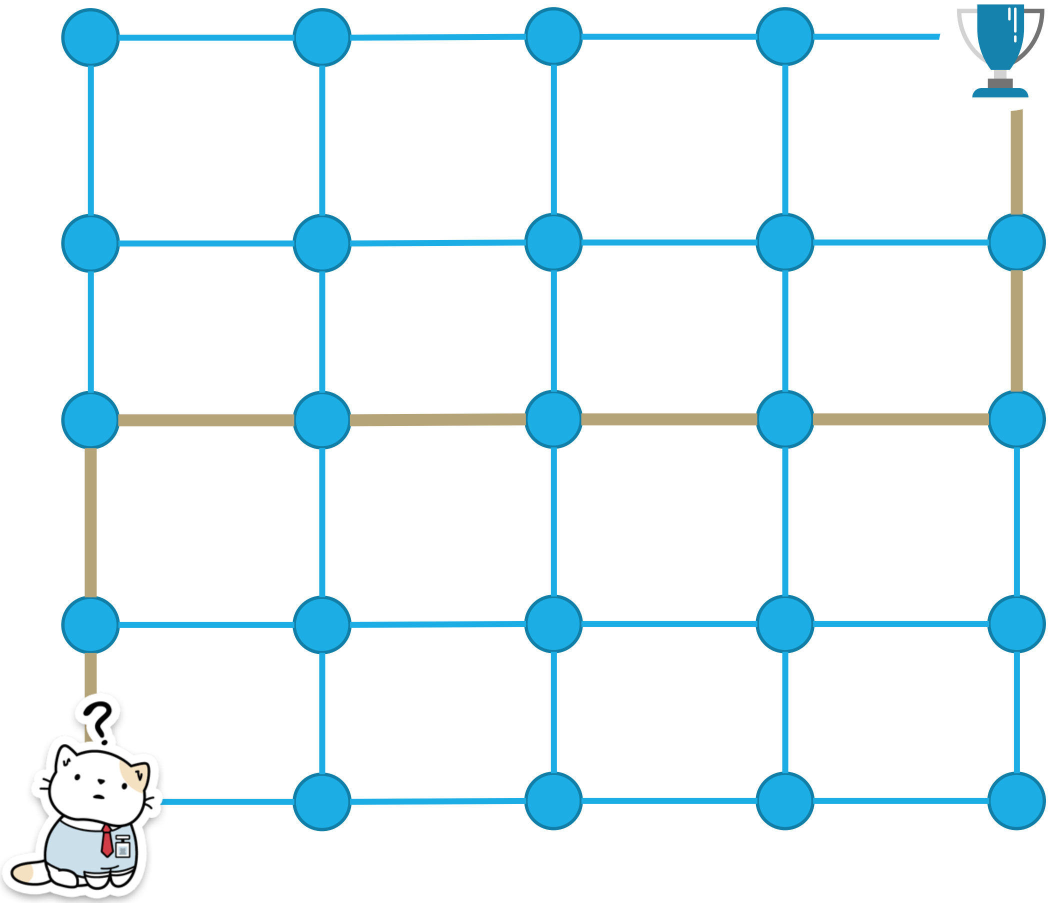
Move 8:

Path Counting



Go from lower-left corner to the upper-right corner.

Each step either goes right or up.

We're going to take 8 steps.

Exactly 4 of the steps must be up and the rest to the right to reach the goal
E.g. {1,2,7,8}

Move 1: ↑
Move 2: ↑
Move 3: →
Move 4: →
Move 5: →
Move 6: →
Move 7: ↑
Move 8: ↑

Path Counting

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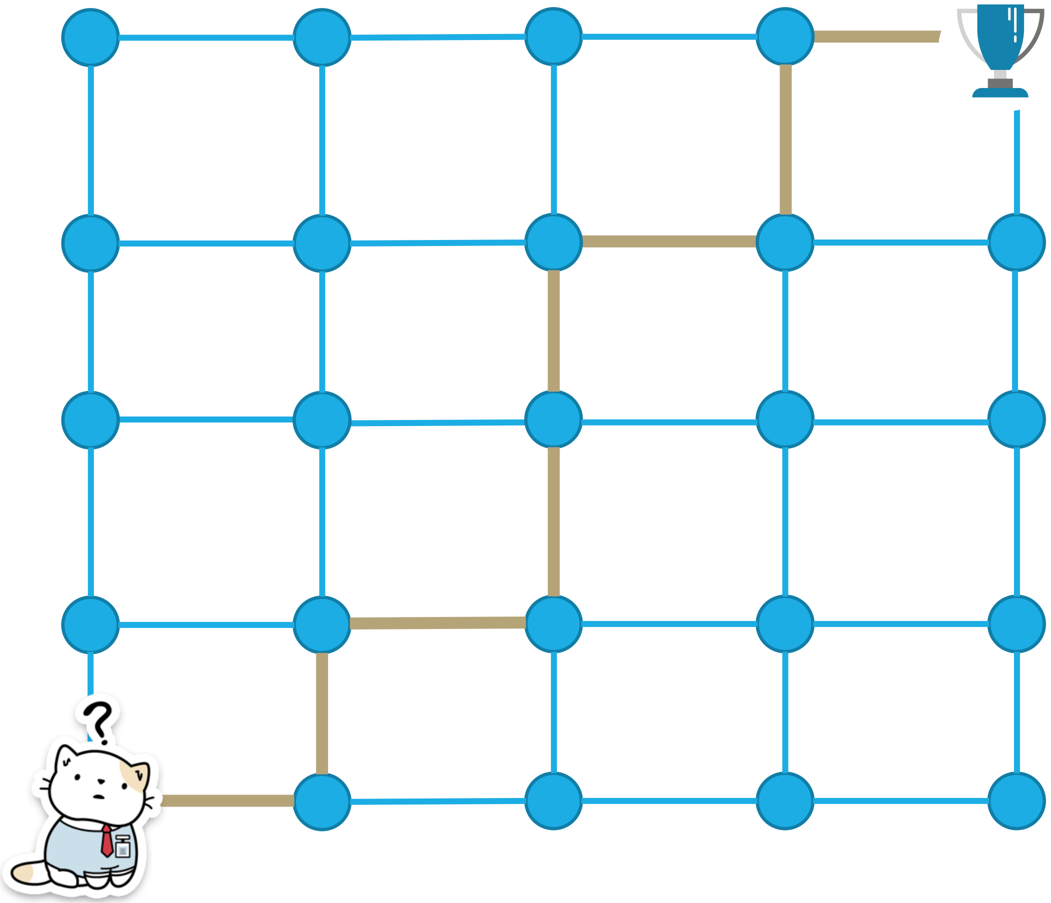

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E.g. {2,4,5,7}

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Move 5: ↑
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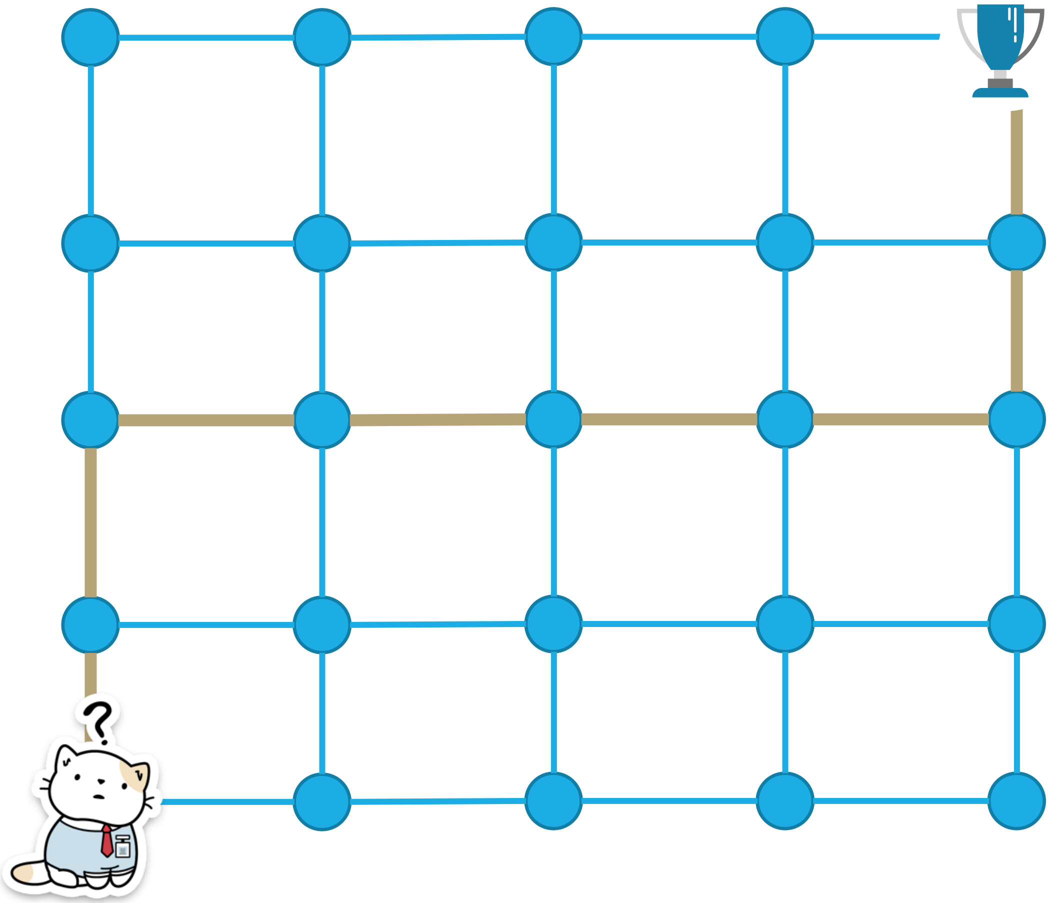
Move 8: →

Path Counting



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We're going to take 8 steps.

Exactly 4 of the steps must be up and the rest to the right to reach the goal

Choose which SET of 4 of the moves will be up (the others will be right).

How many size-4 subsets of $\{1,2,3,4,5,6,7,8\}$ are there?

$\binom{8}{4}$ is the answer.

Binomial Theorem

another application of combinations!

Binomial Theorem

In high school you might have memorized

$(x + y)^2 = x^2 + 2xy + y^2$

And $(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

The Binomial Theorem tells us what happens for every n:

The Binomial Theorem

$$(x + y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$$

$$(x + y)^n = (x + y)(x + y)\ (x + y)\ldots(x + y)$$

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$$y^n + \binom{n}{1} x^1 y^{n-1} + \binom{n}{2} x^2 y^{n-2} + \cdots + x^n$$

Some intuition

The Binomial Theorem

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$$

Intuition: Every monomial on the right-hand-side has either x or y from each of the terms on the left.

How many copies of $x^i y^{n-i}$ do you get? Well how many ways are there to choose i x's and $n-i$ y's? $\binom{n}{i}$.

Formal proof? Induction!

So What?

Well…if you saw it before, now you have a better understanding now of why it's true.

There are also a few cute applications of the binomial theorem to proving other theorems (usually by plugging in numbers for x and y) – you'll do one on HW1.

For example, set $x = 1$ and $y = 1$ then

$2^n = (1 + 1)^n = \sum_{i=0}^{n} \binom{n}{i} 1^i 1^{n-i} = \sum_{i=0}^{n} \binom{n}{i}$.

i.e. if you sum up binomial coefficients, you get 2^n.

What if there are some duplicate/identical items and the rest are distinct?

Rearranging **Distinct Elements**

How many anagrams are there of COMPUTER?

(an anagram is a rearrangement of letters).

There are $8!$ ways to rearrange these 8 distinct letters

Factorial – there are 8 options for the first letter, 7 for the second, etc.

Rearranging **Distinct Elements**

How many anagrams are there of COMPUTER?

(an anagram is a rearrangement of letters).

Rearranging **With Duplicates**

How many anagrams are there of **SEATTLE**?

(an anagram is a rearrangement of letters).

Maybe **7!** we are rearranging 7 letters?

That treats all 7 letters as distinct and counts SEATTLE, SEATTLE as different things!

I swapped the Es (or maybe the Ts)

Rearranging **With Duplicates** (approach 1)

How many anagrams are there of **SEATTLE**?

1. **Pretend all the letters are distinct** (e.g., E_1 different from E_2)

   How many arrangements of SE_1A$\underline{T}_1\underline{T}_2LE_2$? 7!

2. **Divide out overcounting**. Each distinct anagram (e.g., SEATTLE) is counted $2! \cdot 2!$ times because of ordering the E's and ordering the T's

   Divide by $2! \cdot 2!$ so that each distinct anagram is counted exactly once

Final answer $\frac{7!}{2! \cdot 2!}$

Rearranging **With Duplicates** (approach 2)

How many anagrams are there of **SEATTLE**?

1. **Pick positions for the 2 E's.** $\binom{7}{2}$ because the E's are identical
   (e.g., {2, 5} -> __ E __ __ E __ __)
2. **Pick positions for the 2 T's.** $\binom{5}{2}$ because the T's are identical
   (e.g., {1, 3} -> T E T __ E __ __)
3. **Pick positions for the remaining distinct letters.** $3! = 3 \cdot 2 \cdot 1$
   3 options for position for S, 2 options for the A, 1 option for the L
   (e.g., 4, 7, 6 -> T E T S E L A)

$$\binom{7}{2} \cdot \binom{5}{2} \cdot 3 \cdot 2 \cdot 1 = \frac{7!}{5!\,2!} \cdot \frac{5!}{3!\,2!} \cdot 3! = \frac{7!}{2!\,2!}$$

Rearranging **With Duplicates**

What we're doing here is something can be done with *multinomial coefficients*. We won't cover that directly in this class, but feel free ask to me about / look it up on your own time!

Regardless, you should understand the approach taken in this problem :)

Principle of Inclusion-Exclusion

Recall: Sum Rule

How many options do I have for lunch?

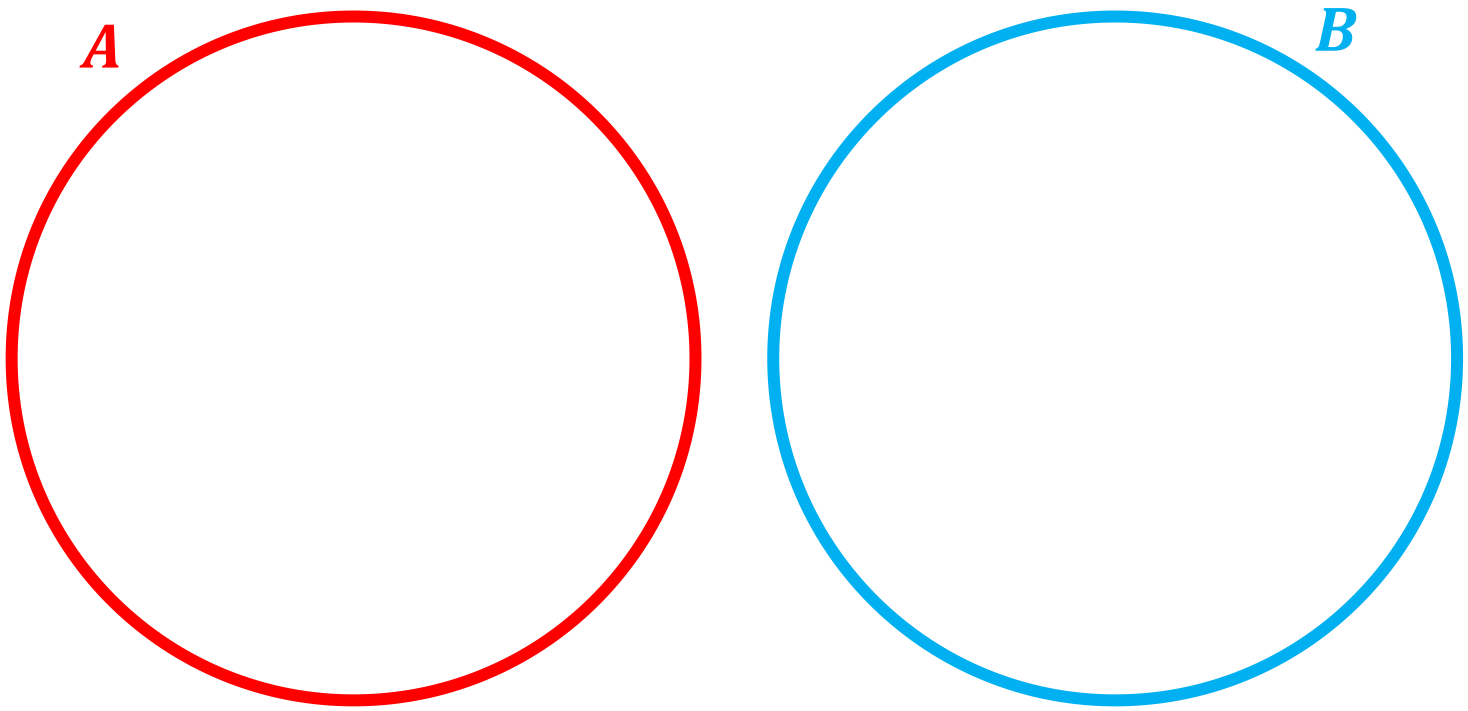
I could go to **Delfino's where there are 6 pizzas** I choose from, **or** I could go to **Supreme where there are 4 pizzas** I choose from (and **none of them are the same between the** two).

How many total choices?

$6 + 4 = 10$

Sum Rule: If you are choosing one thing between n options in one group and m in another group with no overlap, the total number of options is: $n + m$.

Recall: Sum Rule

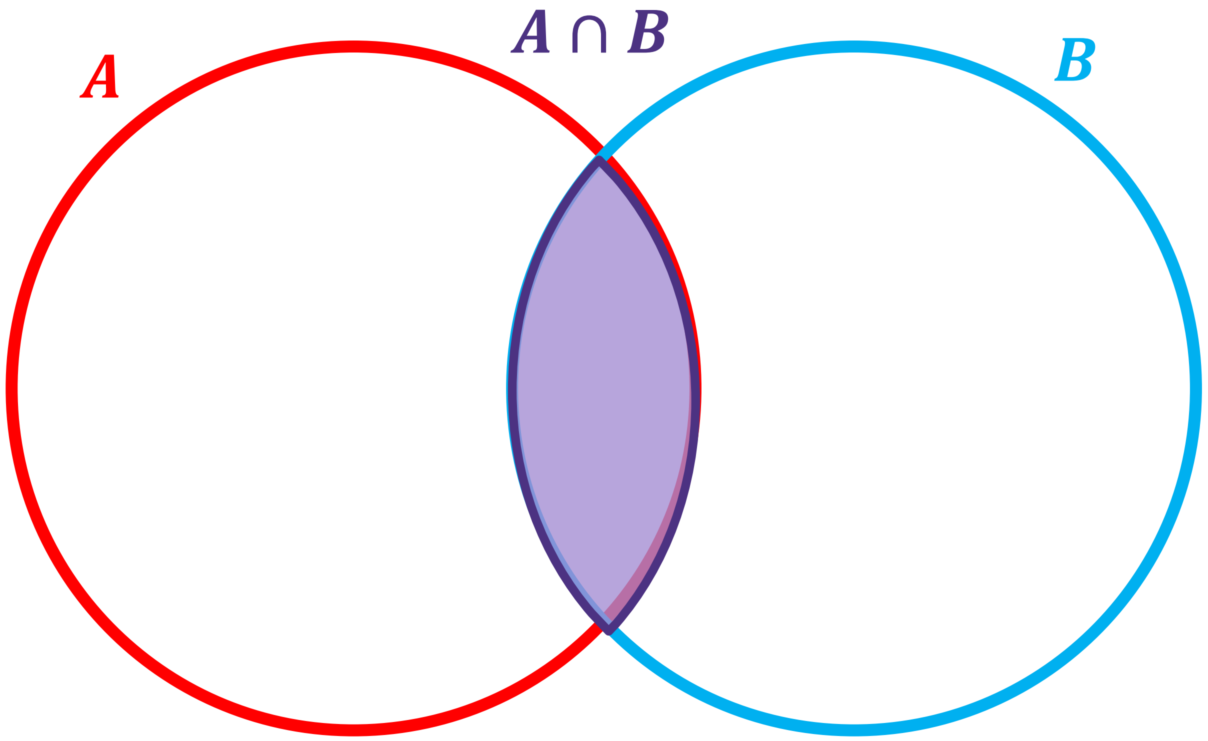


A ~ set of pizzas from Delfino’s → $|A| = 6$

B ~ set of pizzas from Supreme → $|B| = 4$

$|A \cup B| = |A| + |B| = 10$ **because $\boldsymbol{A}$** and **$\boldsymbol{B}$** do not overlap (i.e., $|A \cap B| = 0$)

What if the sets overlap?



But what if there are some pizzas that are sold in **both** Delfino's and Supreme?

i.e., what if A and B overlap and $|A \cap B| > 0$?

A ~ set of pizzas from Delfino’s → $|A| = 6$

B ~ set of pizzas from Supreme → $|B| = 4$

$A \cap B$ ~ set of pizzas in both Delfino’s and Supreme → $|A \cap B| = 2$

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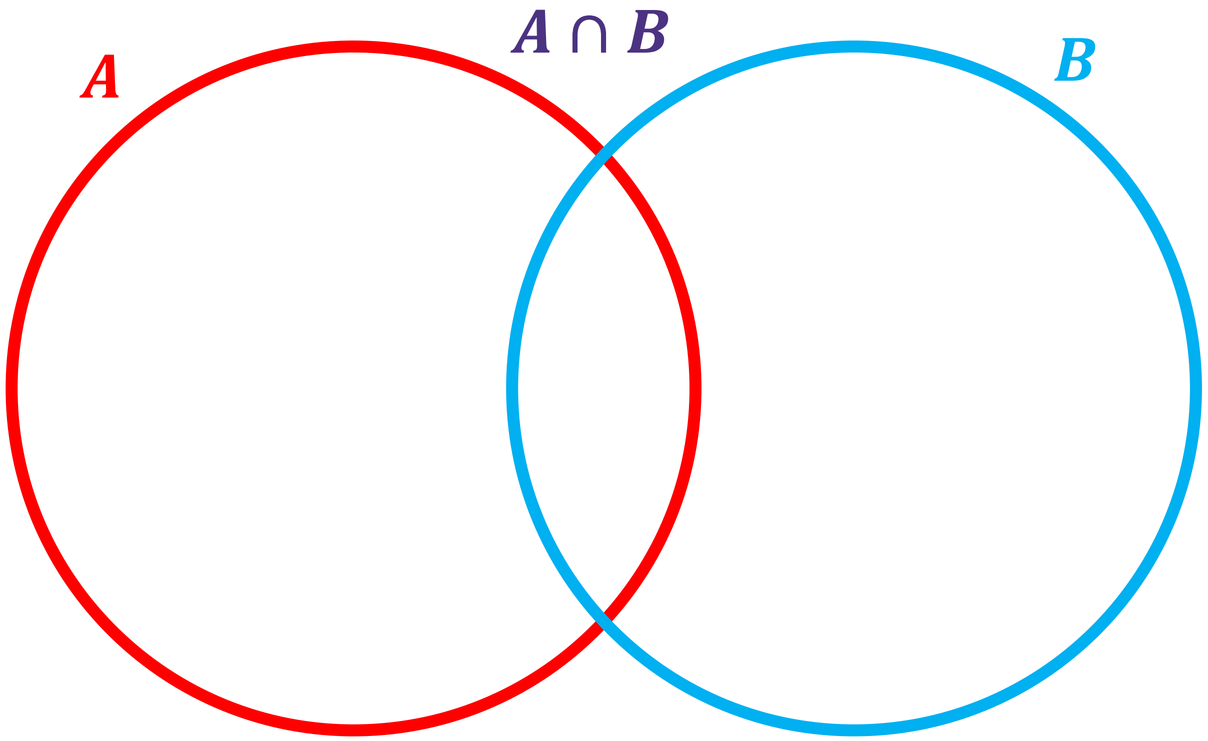

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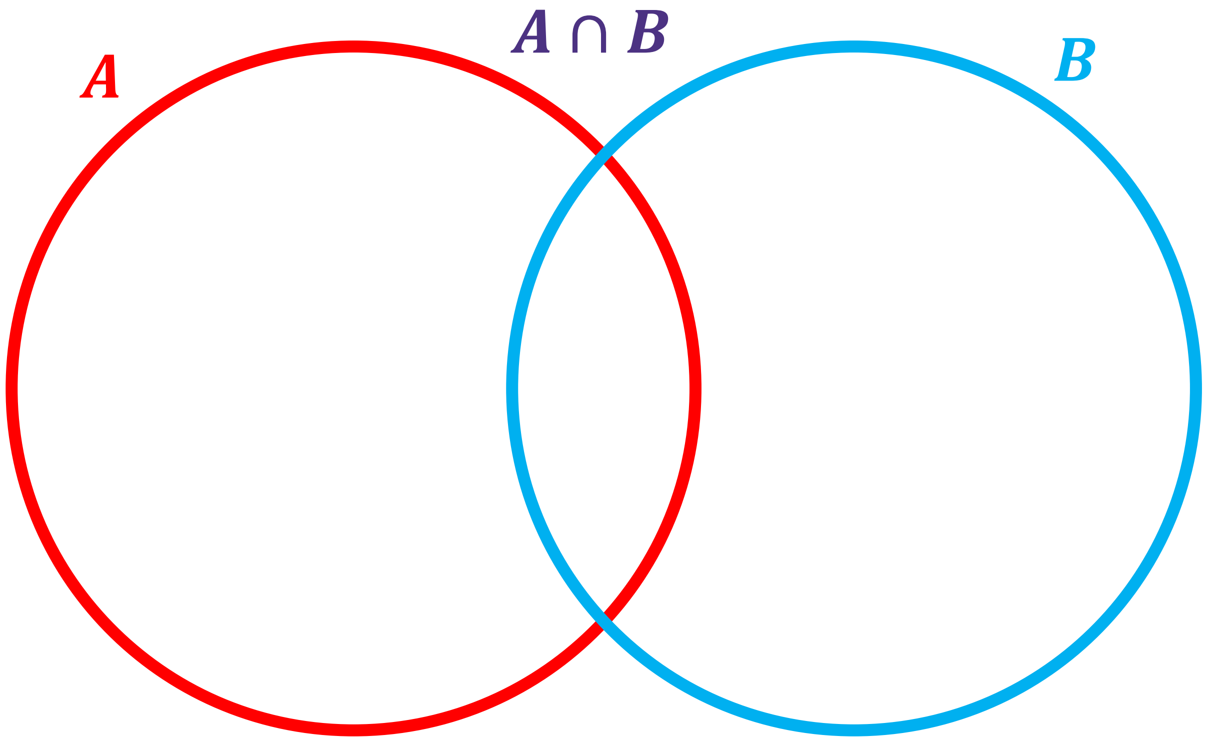

Let's start by adding the options from Delfino's and Supreme… $6 + 4$

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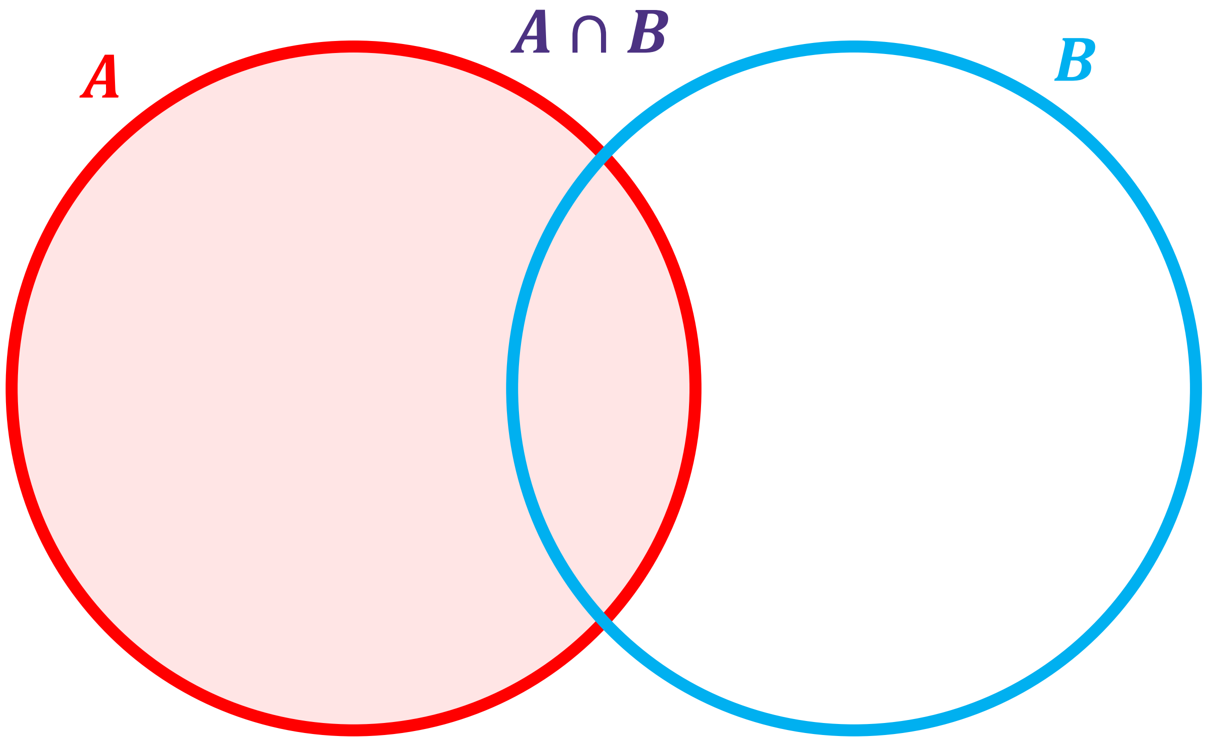

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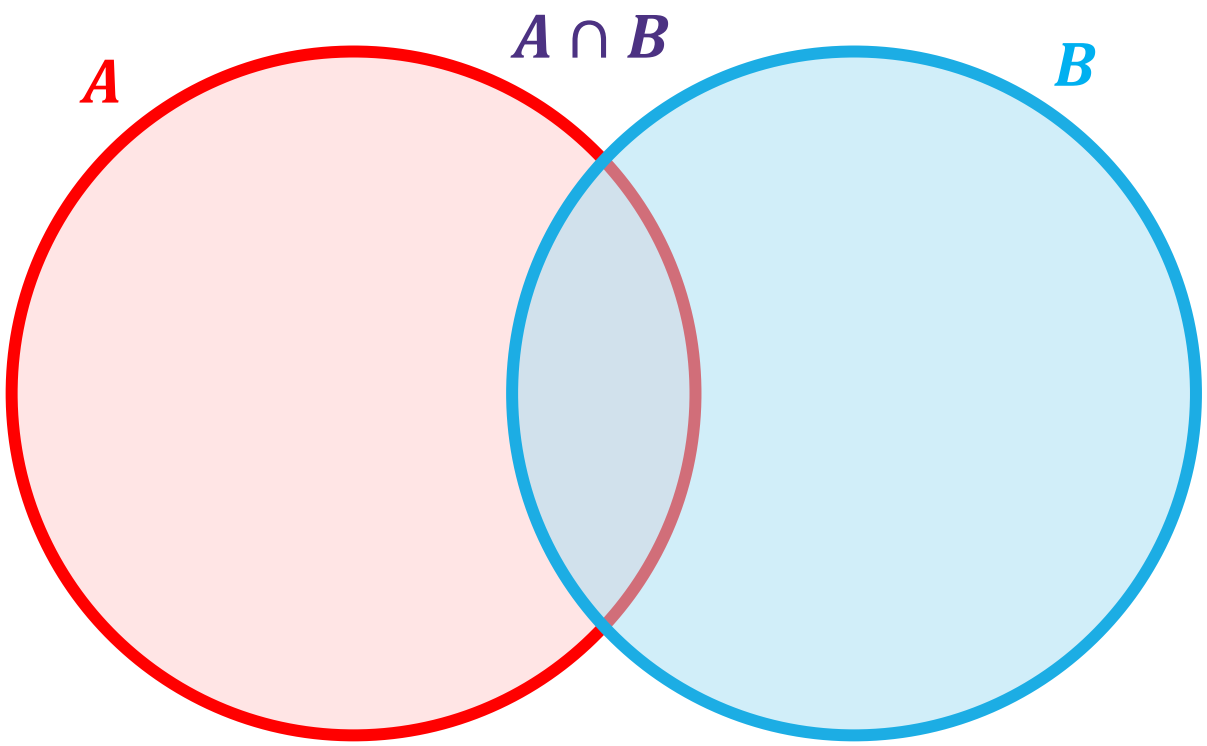

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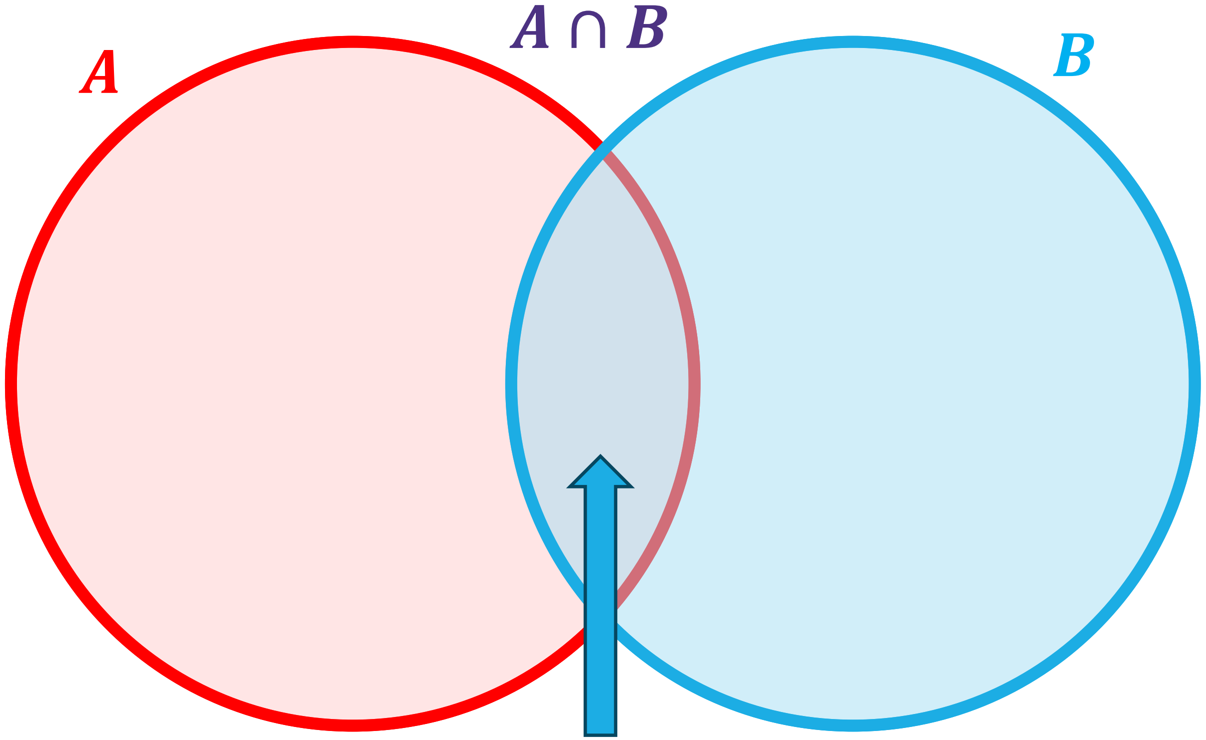

This region $\boldsymbol{A \cap B}$ has been counted twice!

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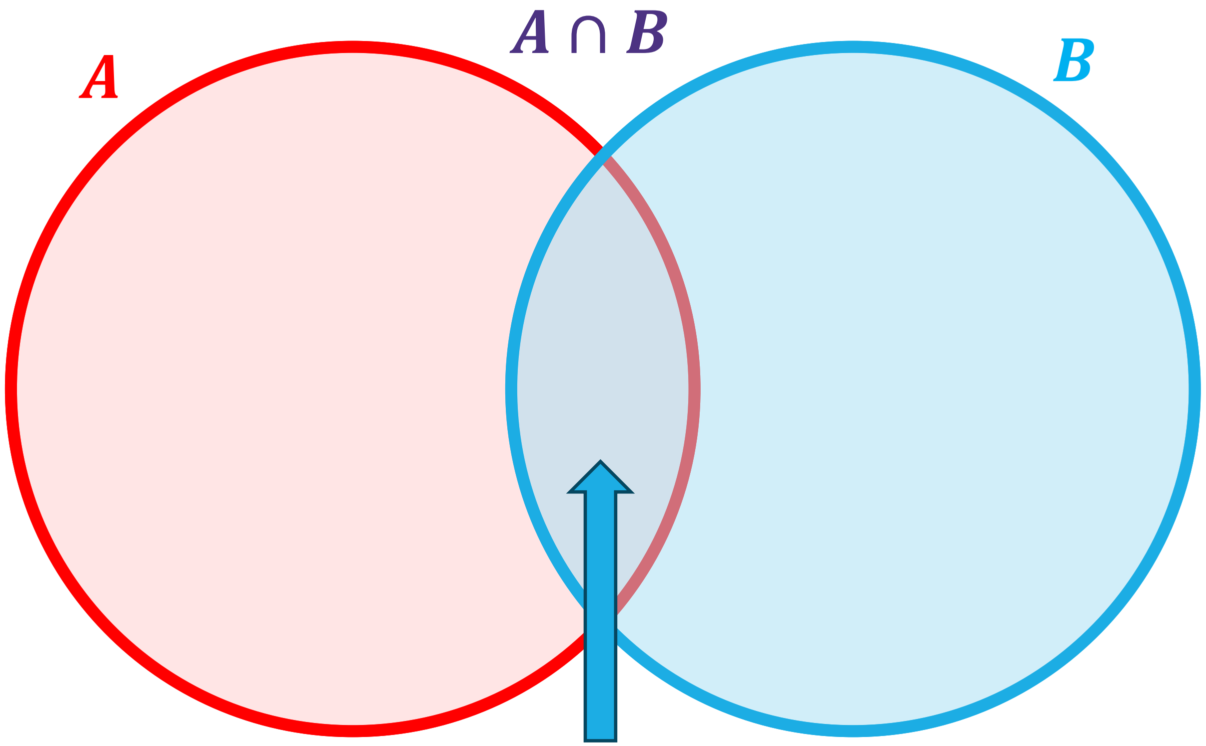

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Let's start by adding the options from Delfino's and Supreme… $6 + 4$

Then we subtract out $|A \cap B| = 2$ because we want to region counted exactly once.

$$|\boldsymbol{A \cup B}| = \mathbf{6 + 4 - 2}$$

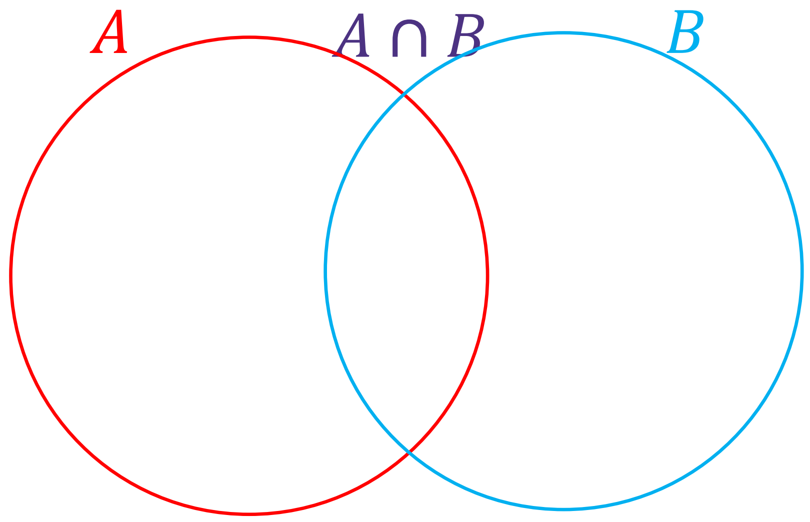
Principle of Inclusion-Exclusion

The sum rule says when A and B are disjoint (no intersection), then $|A \cup B| = |A| + |B|$.

What about when A and B aren't disjoint?

Principle of Inclusion-Exclusion

For any two sets A and B:
$|A \cup B| = |A| + |B| - |A \cap B|$



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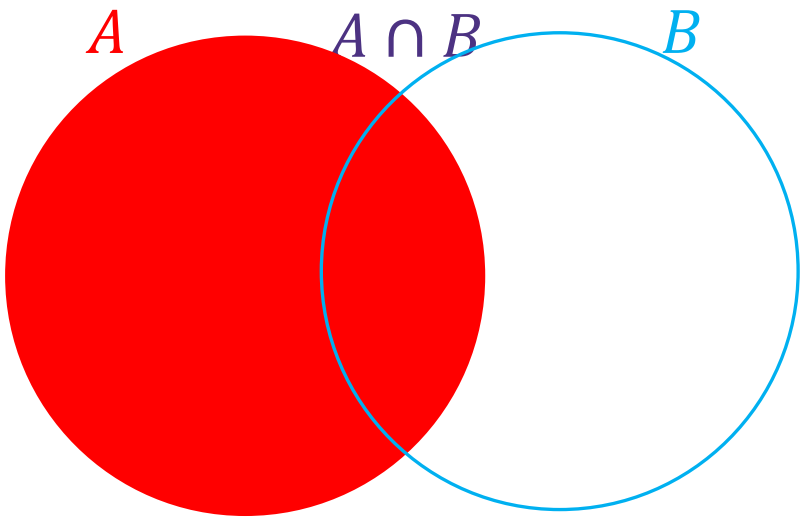

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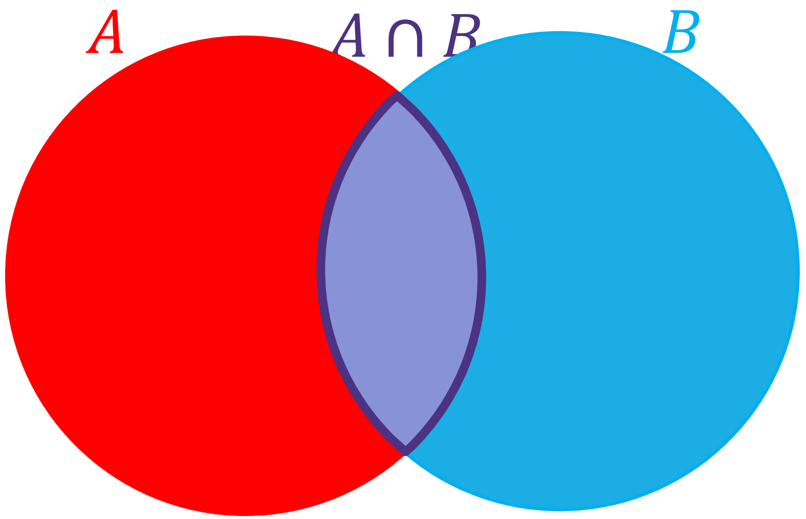

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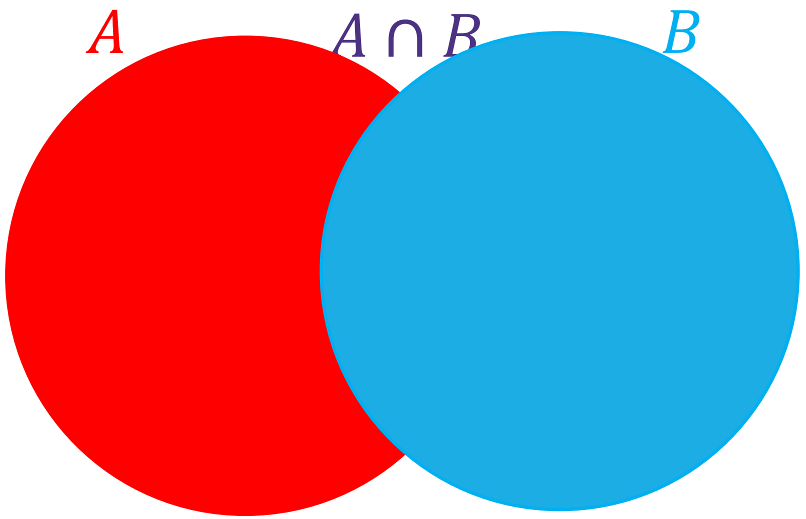
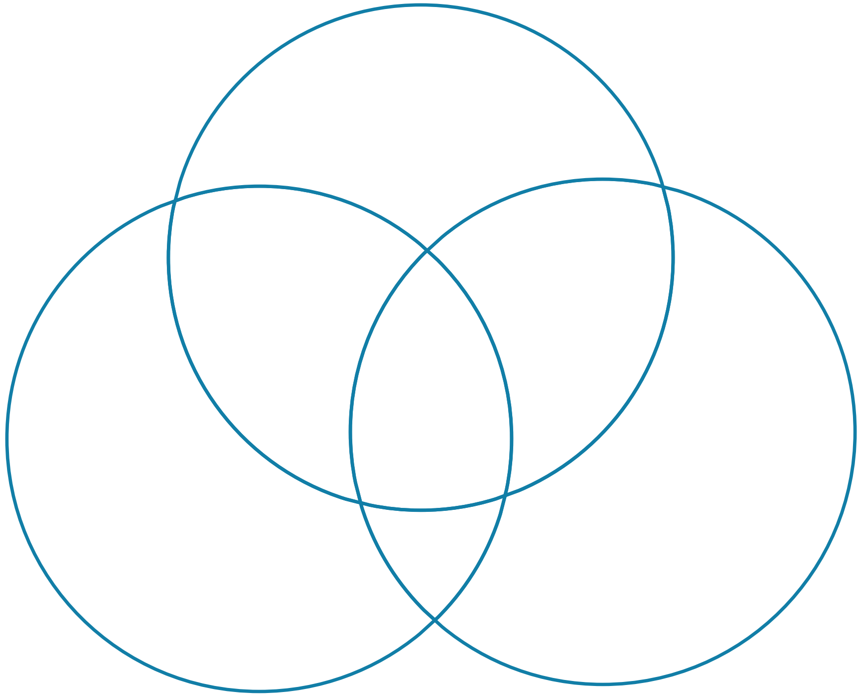


Principle of Inclusion-Exclusion

For three sets:

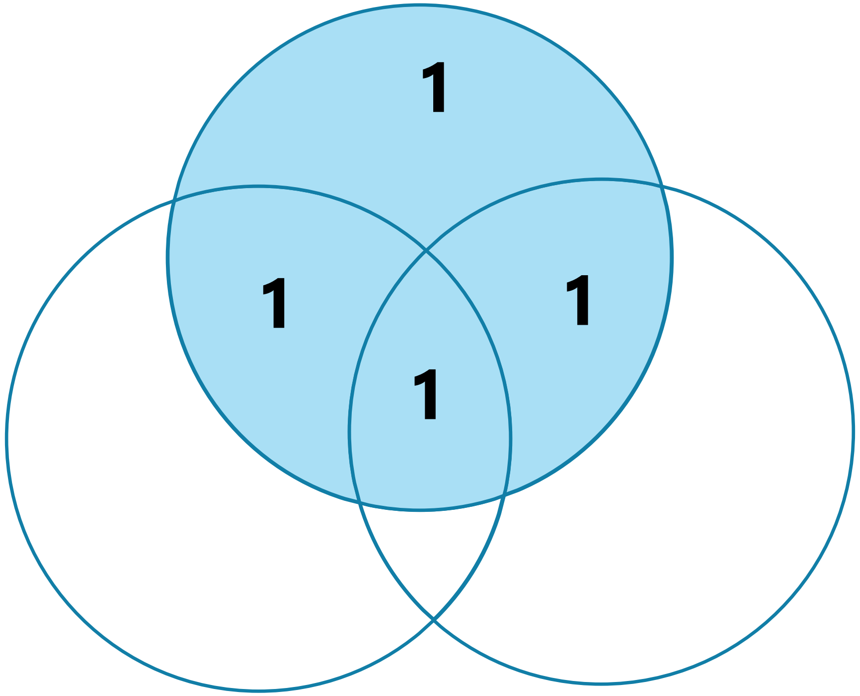
$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$$



Principle of Inclusion-Exclusion

For three sets:

$$|A \cup B \cup C| = \boxed{|A|} + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$$



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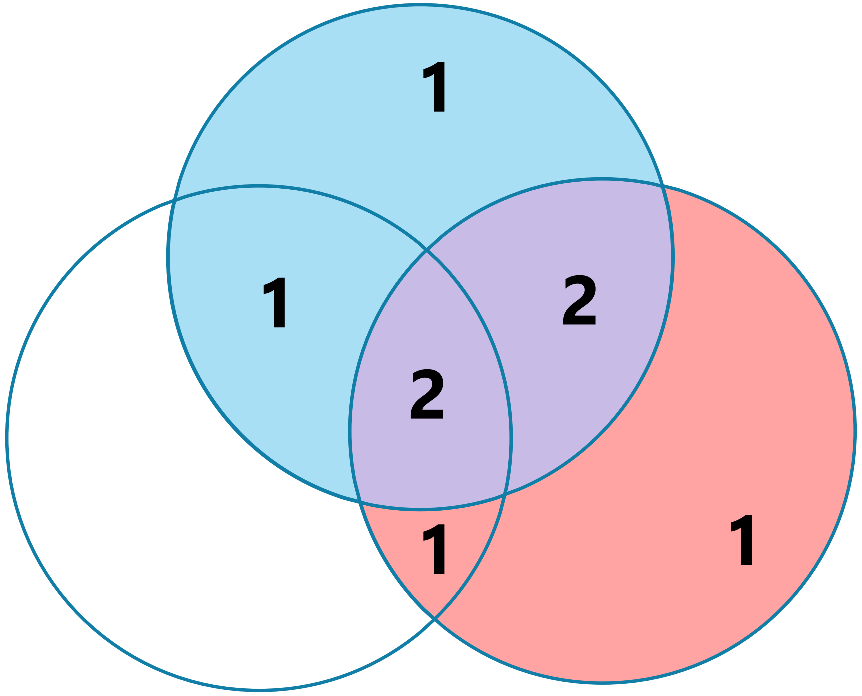

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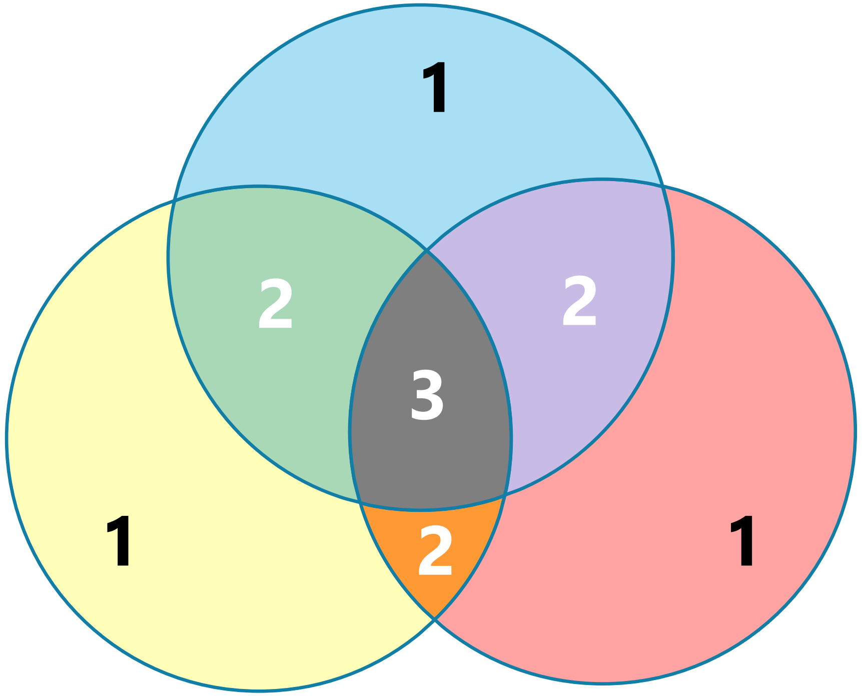

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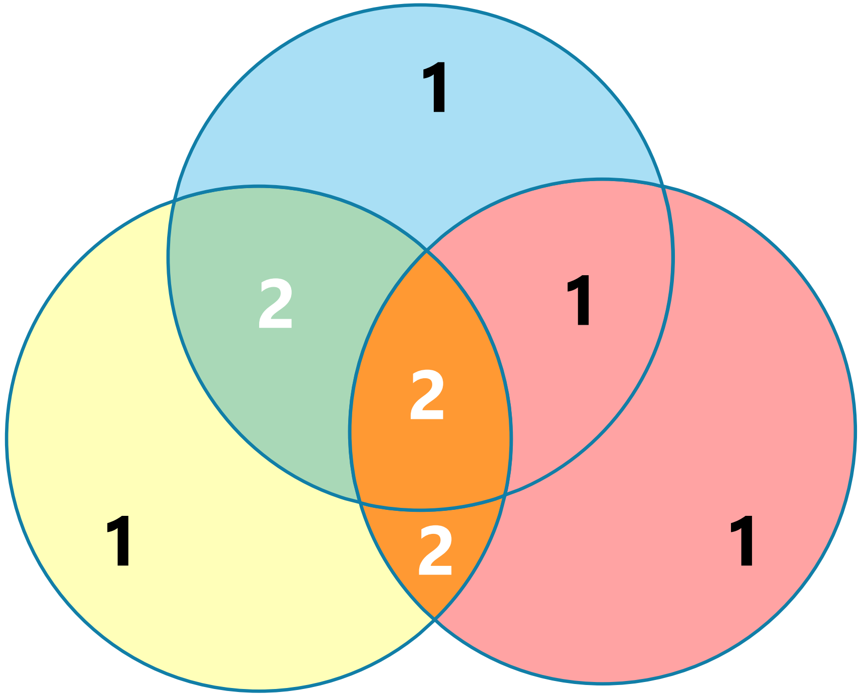
For three sets:

$$|A \cup B \cup C| = \boxed{|A|} + \boxed{|B|} + \boxed{|C|} - \boxed{|A \cap B|} - |B \cap C| - |A \cap C| + |A \cap B \cap C|$$



Principle of Inclusion-Exclusion

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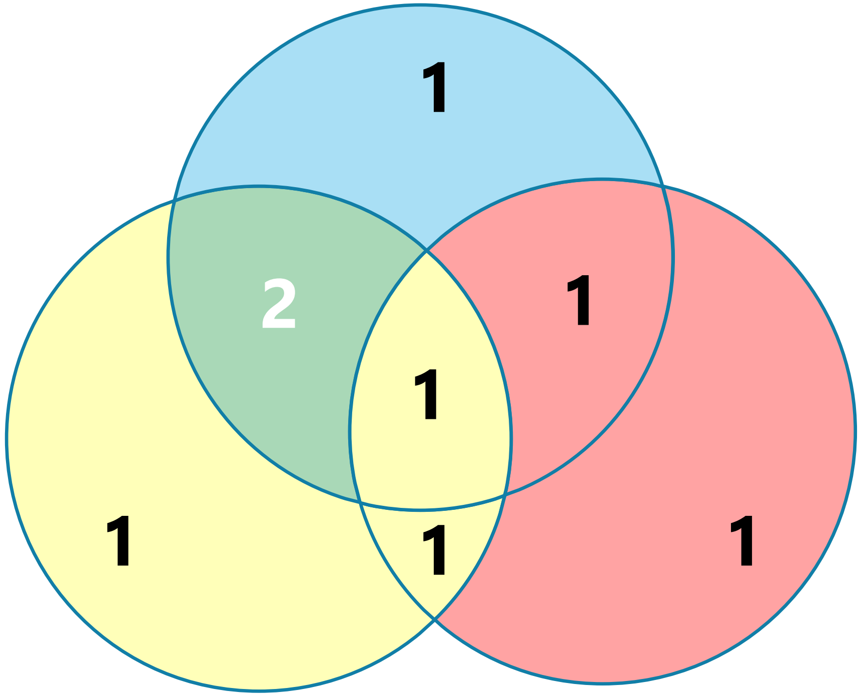

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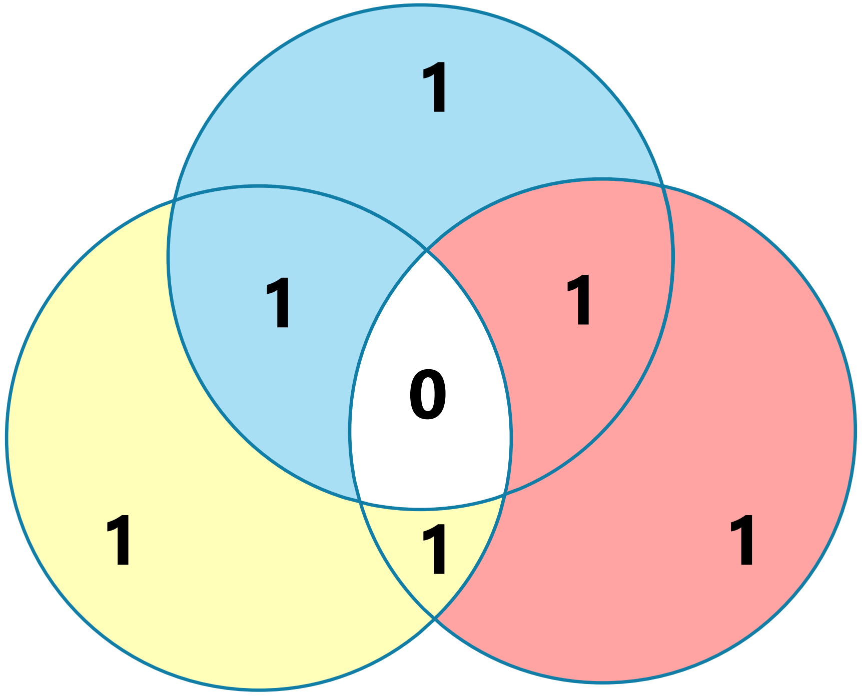

Principle of Inclusion-Exclusion

For three sets:

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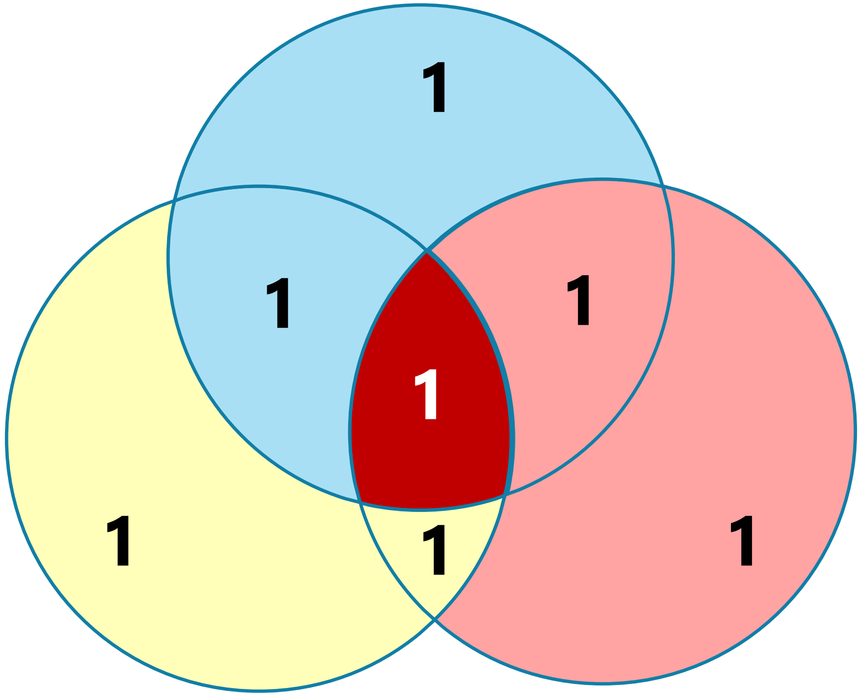

In general:

$$|A_1 \cup A_2 \cup \cdots \cup A_n| =$$
$$|A_1| + |A_2| + \cdots + |A_n|$$
$$-(\,|A_1 \cap A_2| + |A_1 \cap A_3| + \cdots + |A_1 \cap A_n| + |A_2 \cap A_3| + \cdots + |A_{n-1} \cap A_n|)$$
$$+(\,|A_1 \cap A_2 \cap A_3| + \cdots + |A_{n-2} \cap A_{n-1} \cap A_n|)$$
$$- \;\ldots$$
$$+(-1)^{n+1}|A_1 \cap A_2 \cap \cdots \cap A_n|$$

Add the individual sets, subtract all pairwise intersections, add all three-wise intersections, subtract all four-wise intersections,…, [add/subtract] the n-wise intersection.

Principle of Inclusion-Exclusion

Principle of Inclusion-Exclusion

For any two sets A and B:
$|A \cup B| = |A| + |B| - |A \cap B|0$

For three sets:
$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$

For any number of sets:
$|A \cup B \cup C \cup \cdots| =$ singles – doubles + triples – quadruples + ...

Example

How many length 5 strings over the alphabet $\{a, b, c, \ldots, z\}$ contain:

Exactly 2 ‘a’s OR

Exactly 1 ‘b’ OR

No ‘x’s

For what A, B, C do we want $|A \cup B \cup C|$?

A ~ set of length 5 strings with exactly 2 a’s

B ~ set of length 5 strings with exactly 1 b

C ~ set of length 5 strings with no x’s

$A \cup B \cup C$ ~ set of strings that fall into at least one of the above sets

^ we are interested in the size of this set!

$A = \{\text{length 5 strings that contain exactly 2 'a's}\}$

$B = \{\text{length 5 strings that contain exactly 1 'b's}\}$

$C = \{\text{length 5 strings that contain no 'x's}\}$

$$|A \cup B \cup C| = \boxed{|A| + |B| + |C|}\boxed{- |A \cap B| - |B \cap C| - |A \cap C|}\boxed{+ |A \cap B \cap C|}$$

$|A| =$

$|B| =$

$|C| =$

$|A \cap B| =$

$|A \cap C| =$

$|B \cap C| =$

$|A \cap B \cap C| =$

$A = \{$length 5 strings that contain exactly 2 'a's$\}$
$B = \{$length 5 strings that contain exactly 1 'b's$\}$
$C = \{$length 5 strings that contain no 'x's $\}$

$$|A \cup B \cup C| = \boxed{|A| + |B| + |C|}\boxed{- |A \cap B| - |B \cap C| - |A \cap C|}\boxed{+ |A \cap B \cap C|}$$

$|A| = \binom{5}{2} \cdot 25^3$ (choose 'a' spots, remaining chars)
$|B| = \binom{5}{1} \cdot 25^4$ (choose 'b' spot, remaining chars)
$|C| = 25^5$ (select [non-'x'] characters for each spot)

$|A \cap B| =$
$|A \cap C| =$
$|B \cap C| =$

$|A \cap B \cap C| =$

A = {length 5 strings that contain exactly 2 ‘a’s}
B = {length 5 strings that contain exactly 1 ‘b’s}
C = {length 5 strings that contain no ‘x’s }

$$|A \cup B \cup C| = \boxed{|A| + |B| + |C|}\boxed{- |A \cap B| - |B \cap C| - |A \cap C|}\boxed{+ |A \cap B \cap C|}$$

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$|B| = \binom{5}{1} \cdot 25^4$ (choose ‘b’ spot, remaining chars)
$|C| = 25^5$ (select [non-‘x’] characters for each spot)

$|A \cap B| = \binom{5}{2} \cdot \binom{3}{1} \cdot 24^2$ (choose ‘a’ spots, ‘b’ spot, remaining chars)
$|A \cap C| = \binom{5}{2} \cdot 24^3$ (choose ‘a’ spots, remaining [non-‘x’] chars)
$|B \cap C| = \binom{5}{1} \cdot 24^4$

$|A \cap B \cap C| =$

$A = \{$length 5 strings that contain exactly 2 'a's$\}$
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A = {length 5 strings that contain exactly 2 ‘a’s}
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$$|C| = 25^5$$

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$$|B \cap C| = \binom{5}{1} \cdot 24^4$$

$$|A \cap B \cap C| = \binom{5}{2} \cdot \binom{3}{1} \cdot 23^2$$

$$|A \cup B \cup C|$$
$$= \binom{5}{2} \cdot 25^3 + \binom{5}{1} \cdot 25^4 + 25^5 - \binom{5}{2} \cdot \binom{3}{1} \cdot 24^2 - \binom{5}{2} \cdot 24^3 - \binom{5}{1} \cdot 24^4 + \binom{5}{2} \cdot \binom{3}{1} \cdot 23^2$$
$$= 10{,}076{,}470$$

Practical tips

How do I know I'm looking for the size of a union of some sets?

"how many ways for A **or** B **or** C to happen" → $|A \cup B \cup C|$

"how many ways for **at least one** of A, B, or C" → $|A \cup B \cup C|$

"how many ways for **none** of A, B, C" → $\bar{A} \cap \bar{B} \cap \bar{C} = \overline{A \cup B \cup C}$

maybe use complementary counting + inclusion-exclusion (total – # at least 1 occur)

Practical tips

- Give yourself clear definitions of A, B, C.
- Make a list of all the formulas you need before you start actually calculating
- Calculate "size-by-size" and incorporate into the total.

- **Basic check**: If (in an intermediate step) you ever:
  1. Get a negative value
  2. Calculate that $|A \cup B| > |A|$

Then something has gone wrong! Recheck calculations ☺

Summary

Permutations (order matters) and **Combinations** (order doesn't matter)

Applications of Combinations

Path Counting

Binomial Theorem – useful

A useful trick for counting is to pretend order matters, then account for the overcounting at the end (by dividing out repetitions)

More examples

Putting it all together!

Cards

A lot of counting problems deal with cards!

A "standard" deck of cards has 52 cards ($13 \cdot 4 = 52$).

Each card has one of **4 suits**

diamonds ♦,

hearts ♥,

clubs ♧,

spades ♠

and one of **13 values/ranks** (Ace,2,3,4,5,6,7,8,9,10,Jack,Queen, King).

e.g., **Ace♦, 5♦, 5♧, 10♠** are all possible cards

A "k-card-hand" is an unordered set of k cards

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A "k-card-hand" is an unordered set of k cards

How many five-card "flushes" are there? – a flush is a hand of cards all of the same suit. (e.g., {A♠, 3♠, 5♠, 6♠, Q♠})

Five-card "flushes"

How many five-card "flushes" are there? – a flush is a hand of cards all of the same suit.

Think: How would I create a set of cards that is a flush?

Way 1:

1. Pick the suit (e.g., ♠) – $\binom{\mathbf{4}}{\mathbf{1}}$

2. Pick the specific values/cards from that suit (e.g., {A,3,5,6,Q}) - $\binom{\mathbf{13}}{\mathbf{5}}$

Now we've created an unordered 5-card flush! (e.g., {A♠, 3♠, 5♠, 6♠, Q♠})

$$\boxed{\binom{4}{1} \cdot \binom{13}{5}}$$

Five-card "flushes"

Way 2:

Pretend order matters.

1. Pick any first card – 52 options

2. All remaining cards must be from the same suit of that first suit:

    12 options for the 2nd card, 11 options for the 3rd card, etc.

Divide out the overcounting - divide by $5!$, since order isn't supposed to matter (i.e., only count each unordered flush once)

$$\boxed{\frac{52 \cdot 12 \cdot 11 \cdot 10 \cdot 9}{5!}}$$

This equals the same number as what we got on the last slide!

How many 5-card hands have at least 3 aces?

There are 4 Aces (and 48 non aces) in a deck of cards

1. **Choose 3 aces**: $\binom{4}{3}$

2. Then **pick 2 of the 49 remaining cards** to form a 5(the last ace is allowed as well, because we're allowed to have all 4): $\binom{49}{2}$

$\binom{4}{3} \cdot \binom{49}{2}$

What's wrong with this calculation? Does it,

A) Overcount B) Undercount C) It's correct! D) I have no idea :)

Go to pollev.com/cse312

Sleuth's Criterion
How to check if we counted correctly?

For each outcome that we want to count, there should be exactly one set of choices in the sequential process that will lead to that outcome.

> If there are no sequence of choices that will lead to the outcome, we have **undercounted**.

> If there is more than one sequence of choices that will lead to the outcome, we have **overcounted**.

Sleuth's Criterion (in context)

How to check if we counted correctly?

For each "5-card hands with at least 3 aces" that we want to count, there should be <u>exactly one</u> set of choices in the sequential process that will lead to that outcome.

> If there are <u>no sequence of choices</u> that will lead to a particular 5-card hand with at least 3 aces, we have **undercounted**.

> If there is <u>more than one sequence of choices</u> that will lead to a particular **5-card hand with at least 3 aces**, we have **overcounted**.

How many 5-card hands have <u>at least</u> 3 aces?

For each "5-card hands with at least 3 aces" that we want to count, there should be <u>exactly one</u> set of choices in the sequential process that will lead to that outcome.

A♧, A♠, A♦, Q♥, K ♠ is a valid outcome should counted exactly once.

Step 1 (choose 3 aces): {A♧, A♠, A♦}
Step 2 (pick 2 of remaining 49): {Q♥, K ♠}

Great! There's no other set of choices that will lead to this hand.

How many 5-card hands have at least 3 aces?

For each "5-card hands with at least 3 aces" that we want to count, there should be exactly one set of choices in the sequential process that will lead to that outcome.

A♧, A♠, A♦, A♥, K ♠ is a valid outcome should counted exactly once.
But…

Step 1 (choose 3 aces): {A♧, A♠, A♦}
Step 2 (pick 2 of remaining 49): {A♥, K ♠}

Step 1 (choose 3 aces): {A♧, A♥, A♦}
Step 2 (pick 2 of remaining 49): {A ♠, K ♠}

Both of these are different choices in the sequential process and are counted separately, but they are the same hand!
This is overcounting ☹

Fixing The Overcounting

How many 5-card hands have at least 3 aces?

Way 1: We could start with our incorrect solution & **subtract the overcounting.**

Our original incorrect solution:
1. **Choose $\mathbf{3}$ aces:** $\binom{4}{3}$, 2. Then **pick $\mathbf{2}$ of the $\mathbf{49}$ remaining cards:** $\binom{49}{2}$ → $\binom{4}{3} \cdot \binom{49}{2}$

What kinds of hands do we overcount (counted many times in the sequential process)?

> 5-card hands with 4 Aces (i.e., a hand like {A♧, A♠, A♦, A♥, X})

> This hand is counted 4 different times (each row below is a different set of choices)

{A♧, A♠, A♦}, {A♥, X}

{A♧, A♠, A♥}, {A♦, X}

{A♧, A♥, A♦}, {A♠, X}

{A♥, A♠, A♦}, {A♧, X}

How many 5-card hands have at least 3 aces?

Way 1: We could start with our incorrect solution & **subtract the overcounting.**

Our original incorrect solution:
1. **Choose** $\mathbf{3}$ **aces**: $\binom{4}{3}$, 2. Then **pick** $\mathbf{2}$ **of the** $\mathbf{49}$ **remaining cards**: $\binom{49}{2}$ → $\binom{4}{3} \cdot \binom{49}{2}$

What kinds of hands do we overcount (counted many times in the sequential process)?

> 5-card hands with 4 Aces (i.e., a hand like {A♣, A♠, A♦, A♥, X})

{A♣, A♠, A♦}, {A♥, **X**}
{A♣, A♠, A♥}, {A♦, **X**}
{A♣, A♥, A♦}, {A♠, **X**}
{A♥, A♠, A♦}, {A♣, **X**}

So, how many outcomes are overcounted?

> There are $\binom{4}{4} \cdot 48 = 48$ 5-card hands with all 4 Aces

> Each of these hands is counted 4 times, but we only want to count it once

> So we've counted $(4-1) \cdot 48 = 3 \cdot 48$ processes that shouldn't count.

That would give a corrected total of $\binom{4}{3} \cdot \binom{49}{2} - 3 \cdot 48$

How many 5-card hands have at least 3 aces?

Way 1: We could **subtract out the overcounting** - count exactly which hands are overcounted in our sequential process, and how many times each of those hands are overcounted, and subtract that from our initial count.

$$\binom{4}{3} \cdot \binom{49}{2} - 3 \cdot 52$$

Way 2: Try a different approach! The problem with our original solutions was trying to account for the "at least" - **come up with disjoint sets and count separately.**

Case 1: There are exactly 3 aces: $\binom{4}{3} \cdot \binom{48}{2}$

Case 2: There are exactly (all) 4 aces: $\binom{4}{4} \cdot \binom{48}{1}$

Applying the sum rule: $\binom{4}{3} \cdot \binom{48}{2} + \binom{4}{4} \cdot \binom{48}{1}$

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Applying the sum rule: $\binom{4}{3} \cdot \binom{48}{2} + \binom{4}{4} \cdot \binom{48}{1}$

Does this overcount/undercount?

For a valid outcome, there should be exactly 1 set of choices leading to that outcome:

A♧, A♠, A♦, Q ♧, K ♠ - this will fall under the first case. The only possible set of choices leading to this is {A♧, A♠, A♦} in the 1st step and {Q ♧, K ♠} in the 2nd

How many 5-card hands have at least 3 aces?

Way 2: Try a different approach! The problem with our original solutions was trying to account for the "at least" - **come up with disjoint sets and count separately.**

Case 1: There are exactly 3 aces: $\binom{4}{3} \cdot \binom{48}{2}$

Case 2: There are exactly (all) 4 aces: $\binom{4}{4} \cdot \binom{48}{1}$

Applying the sum rule: $\binom{4}{3} \cdot \binom{48}{2} + \binom{4}{4} \cdot \binom{48}{1}$

Does this overcount/undercount?

For a valid outcome, there should be exactly 1 set of choices leading to that outcome:

A♧, A♠, A♦, A♥, K ♠ - this will fall under the second case. The only possible set of choices leading to this is {A♧, A♠, A♦, A♥} in the 1st step and {K ♠} in the 2nd

Takeaway

It’s hard to count sets where one of the conditions is “at least X”

Depending on the problem, you may have to:

- Break into need to break those conditions up into disjoint sets and use the sum rule

- Take the complement and find the total ways to have <X things
  usually more helpful when we’re asking for ways for “at least 1 thing occur”

- If we want at least 1 of some conditions to be met, may be able to write as a union of sets and use inclusion exclusion

Extra Practice

Books, revisited

Remember the books problem from lecture 1? Books 1,2,3,4,5 need to be assigned to Alice, Bob, and Charlie (each book to exactly one person).

Now that we know combinations, try a sequential process approach. It won't be as nice as the change of perspective, but we can make it work.

Break into cases based on how many books Alice gets, use the sum rule to combine.

Books, revisited

Step 1: give Alice gets 0 books (1 way to do this)

Step 2: give Bob a subset of the remaining books 2^5 ways.

Step 3: give Charlie the remaining books (no choice – 1 way)

$+$

Step 1: give Alice 1 book ($\binom{5}{1}$ ways to do this)

Step 2: give Bob a subset of the 4 remaining books 2^4 ways.

Step 3: give Charlie the remaining books (no choice – 1 way)

$+ \ \ldots$

Books, revisited

Add all the options together

$$1 \cdot 2^5 \cdot 1 + \binom{5}{1} \cdot 2^4 \cdot 1 + \binom{5}{2} \cdot 2^3 \cdot 1 + \binom{5}{3} \cdot 2^2 \cdot 1 + \binom{5}{4} \cdot 2^1 \cdot 1 + \binom{5}{5} \cdot 2^0 \cdot 1$$

If you plug and chug, you'll get the number we got last time. It took quite a bit of work, but we got there!