Homework 2

Due: Wednesday, July 3, by 11:59pm

Instructions

See the instructions and FAQ for homeworks on the course website for important notes on the submission format!

Solutions submission. You must submit your solution via Gradescope (the Gradescope assignment will be released on Saturday). In particular:

- Submit a *single* PDF containing the solution to all of Tasks 1-6 to Gradescope under "**HW 2 [Written]**".
- Each numbered task should be solved on its own page (or pages). Follow the prompt on Gradescope to link tasks to your pages.
- Do not write your name on the individual pages – Gradescope will handle that.

Task 0 – Collaborators [0 pts]

List the full names of anyone you collaborated with on this homework. If you did not collaborate with anyone, write "None" in this section.

Task 1 – Proofs That Really Count [16 pts]

Prove each of the following with a **combinatorial proof**. An algebraic solution will be marked substantially incorrect - and it will be much harder and not as informative and fun!

See section examples for how much work you should show. When explaining each side of the expression, clearly explain how each term and value count what is being counted. This includes being clear about the values on the bounds of the summation, etc.

a) (8 points)

$$\binom{a+b}{a} = \sum_{i=0}^{a} \binom{a}{i}\binom{b}{i}$$

Hint: Start with the left-hand side and imagine you are choosing a group of a apples from a collection consisting of a red apples and b green apples.

b) (8 points)

$$\sum_{i=0}^{a} \binom{a}{i}\binom{i}{b} = \binom{a}{b}2^{a-b}$$

Hint: Think about choosing students to go on a field trip from a group of a students. Out of the students chosen to go on the field trip, b of them need to be chosen as group leaders.

Task 2 – Where's That Gift? [10 pts]

At a wedding reception, 50 uniquely labeled gift boxes are arranged in a circular display. Each gift box is supposed to be placed in a spot marked with a corresponding label. However, due to a mix-up, none of the gift boxes are placed at their correctly labeled spot initially. Use the pigeonhole principle to show that it is possible to rotate the circular display so that at least two gift boxes are correctly placed. *Be sure to specify precisely what the pigeons are, precisely what the pigeonholes are, and precisely what the mapping of pigeons to pigeonholes is.*

Hint: Think about what property two boxes need to share in order for it to be possible to rotate the display so that both of them end up in the correct place. Try drawing out some examples to help with this.

Task 3 – You're a VIP [12 pts]

a) (4 points) A group of 20 friends are at a music festival with 50 (distinct) performances. Each friend plans to attend exactly 2 performances due to time constraints. How many different ways can they select which performances each person attends? (Any number of people (including 0) can attend each performance)

b) (4 points) While secretly exploring the venue a few days before the festival, you find 25 **identical** VIP passes. How many ways are there are distribute these VIP passes among 10 friend groups attending the festival? Each group can receive any number of VIP passes, including zero.

c) (4 points) (Continuing from part b) To be more fair, you decide that each group will get at least one VIP pass. How many ways are there to distribute these 25 identical VIP passes among the 10 friend groups, such that each group gets at least one pass?

Task 4 – Spicy Chai [10 pts]

You're making a large batch of chai latte that is going to include 6 teaspoons of spices. There are a total of 12 spices available. The order of spices chosen does not matter, and you may repeat spices. We only care about how many teaspoons of each spices are chosen. For example, a valid outcome (that should only be counted once) is 2 teaspoons of cinnamon, 1 teaspoon of anise, and 3 teaspoons of cardamon. How many drinks are possible that have at least one teaspoon of either cardamon or cloves?

[HW continues on next page]

Task 5 – What are The Odds [17 pts]

For each of the following scenarios **first describe the sample space** and **indicate how big it is** (i.e., what its cardinality is) and then answer the question.

Below is an example of a sample problem and solution - the solution here would get full credit for explaining the sample space and its cardinality

In a programming competition, there are 10 problems to solve, 4 of which use CSE 312 knowledge. If a contestant randomly selects 3 problems to solve, what is the probability that at least 2 of the chosen problems involve CSE 312 knowledge?

The sample space Ω is the set of possible unordered sets of 3 problems the contestant chooses. $|\Omega| = \binom{10}{3}$ because we are choosing an unordered subset of 3 problems from the 10. The event is the set of sets of problems that have at least 2 CSE 312 related problems. We use the sum rule splitting into the disjoint cases of having exactly 2 312 problems (we pick 2 of the 4 312 problems, and 1 of the 6 non-312 problems), and exactly 3 312 problems. So, $|E| = \binom{4}{2}6 + \binom{4}{3}$. Because every outcome in the sample space is equally likely, $P(E) = \frac{\binom{4}{2}6+\binom{4}{3}}{\binom{10}{3}}$.

a) (3 points) You flip a fair coin 50 times. What is the probability of exactly 20 heads?

b) (3 points) At a school, there are 10 different extracurricular classes (robotics, crocheting, dance, etc.). Twenty students are randomly assigned to these classes, with each assignment being equally likely. What's the probability that exactly 3 students are assigned to the crocheting class?

c) (4 points) There are 30 psychiatrists and 24 psychologists attending a conference. Three of these 54 people are randomly chosen to take part in a panel discussion.
- What is the probability that at least one psychologist is chosen?
- What is the probability that exactly three psychologists are chosen?

d) (3 points) A fair coin is flipped n times (each outcome in $\{H, T\}^n$ is equally likely). What is the probability that all heads occur at the end of the sequence? (The case that there are no heads is a special case of having all heads at the end of the sequence, i.e. 0 heads.)

e) (4 points) You're making a password generator for your new website that randomly generates 8-character passwords. Each password consists of uppercase letters (A-Z), lowercase letters (a-z), and digits (0-9). Each character in the password is chosen randomly from the total of $26 + 26 + 10 = 62$ characters. What is the probability that a randomly generated password contains exactly 3 uppercase letters, 2 lowercase letters, and 3 digits?

Task 6 – Don't Count Me Out [11 pts]

Consider the question: what is the probability of getting a **7-card** poker hand (order doesn't matter) that contains at least two 3-of-a-kind (3-of-a-kind means three cards of the same rank). For example, this would be a valid hand: ace of hearts, ace of diamonds, ace of spaces, 7 of clubs, 7 of spades, 7 of hearts and queen of clubs. (Note that a hand consisting of all 4 aces and three of the 7s is also valid.) Here is how we might compute this:

> Each of the $\binom{52}{7}$ hands is equally likely. Let E be the event that the hand selected contains at least two 3-of-a-kinds. Then
>
> $$\mathbb{P}(E) = \frac{|E|}{\binom{52}{7}}$$
>
> To compute $|E|$, apply the product rule. First pick two ranks that have a 3-of-a-kind (e.g. ace and 7 in the example above). For the lower rank of these, pick the suits of the three cards. Then for the higher rank of these, pick the suits of the three cards. Then out of the remaining $52 - 6 = 46$ cards, pick one. Therefore
>
> $$|E| = \binom{13}{2} \cdot \binom{4}{3} \cdot \binom{4}{3} \cdot \binom{46}{1} \qquad \text{and hence} \qquad \mathbb{P}(E) = \frac{\binom{13}{2} \cdot 4^2 \cdot 46}{\binom{52}{7}}.$$

a) (3 points) **Explain what is wrong with this solution**. If there is over-counting in $|E|$, characterize what kinds of hands are counted more than once, and how many times each such hand is counted. If there is under-counting in $|E|$, explain what hands are not counted.

b) (4 points) Give the correct answer for $\mathbb{P}(E)$ by **subtracting out or dividing out hands that were over-counted and/or adding hands that were not counted.**

c) (4 points) Give the correct answer for $\mathbb{P}(E)$ by **using a different approach** (e.g., using a different sequential process or splitting into cases).